Range Voting satisfies properties that *no* rank-order system can

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April 28, 2007*

Abstract

The top two reasons people are interested in replacing Plurality Voting with something else are

1. the "vote splitting" problem and
2. the "lesser evil" problem.

Crudely speaking, the properties which cause a voting system to be immune to those two problems are (respectively), ICC=*i*mmunity to *c*andidate *c*loning, and AFB=*a*voids "*f*avorite *b*etrayal." We show that no voting system based on rank-order preference ballots, can enjoy both properties simultaneously. Indeed, with a slight weakening AFB′ of the definition of AFB, we show impossibility of achieving both ICC and AFB′ even by rank-order voting systems *with rank-equalities allowed*.

However, *range voting* satisfies both properties. Range voting is the system where each voter rates each candidate on an 0-9 scale and highest average score wins. At the end we survey some other important voting-system impossibility theorems, again noting that range voting often "accomplishes the impossible."

1 Background

For background on single-winner voting systems see [8, 11].

Voting impossibility theorems are surveyed in [14]c.

Range voting has many remarkable properties. Most important is the experimental (computer Monte Carlo) measurement ([14]a,d[1]) of its *Bayesian Regret* showing that (by the BR yardstick) it robustly and clearly outperforms every other common voting system proposal found in the political science literature. BR is a quantitative measure of voting system quality. The measurements indicate that adopting Range Voting instead of the presently-dominant voting system, plurality voting[2] would yield an social utility-improvement comparable or greater to the improvement that was obtained by inventing democracy. My estimates[3] are that the world suffers 5500 unnecessary deaths per each day's delay in enacting range voting.

It also is interesting that certain social insects, including honeybees, adopted range voting millions of years ago to make collective decisions [14]e.

CRV co-founder Jan Kok pointed out the somewhat non-obvious fact that range voting (with integer scores) can be handled immediately by every voting machine in the USA, with no modification and no reprogramming needed. This is despite the fact that many of these machines are not computerized and were designed only to handle plurality voting. The reason is that, e.g, a C-candidate range voting election can be equivalently regarded, as far as the voting machines are concerned, as $10C$ artificial "plurality elections." (Interactive demo elections which you can participate in are available at the CRV website [1]).

[1] See also http://RangeVoting.org/BayRegDum.html and http://RangeVoting.org/UniqBest.html.

[2] Plurality: Your vote is naming one candidate. The most-named candidate wins.

[3] http://RangeVoting.org/LivesSaved.html

The best source of information on range voting is the Center For Range Voting [1] (CRV) website (which I co-founded) and the author's papers [14].

A range vote is said to be *normalized* if it ranks that voter's favorite 9 and her least-favorite 0. Strategic range votes can always be assumed to be normalized.

2 Proof that range voting satisfies certain criteria that no ranked-ballot voting method can

Theorem (Main result): *These four criteria (or "axioms"), for a nontrivial voting system which inputs pure-rank-order-ballots and outputs the name of a winner (or a set of co-equal winners to be chosen among by random tiebreak), are incompatible:*
b1. AFB = avoids favorite betrayal
b2. ICC = immune to candidate cloning
b3. no vetoer = There does not exist a voter whose vote can single-handedly prevent a candidate of her choice from winning, regardless of how the other voters vote.
b4. neutrality = symmetry under candidate renaming = permuting the candidate names on the ballot rankings permutes their winning probabilities in the same way.

The AFB and ICC criteria will be defined shortly. We shall actually focus on proving the more-complicated-to-state, but easier to prove Equivalent Theorem below, which is based on more axioms. There are actually *many* equivalent theorems, as we shall explain shortly after the theorem statement (and explain why they are equivalent). The Main Theorem arises because its (simpler and fewer) axioms imply the axioms in the Equivalent Theorem. As we shall explain, these implications were shown by previous authors or can quickly be inferred from their results.

Equivalent Theorem: *These six criteria, for a single-winner voting system based on pure-rank-order-ballots, are incompatible:*
a1. AFB = avoids favorite betrayal
a2. ICC = immune to candidate cloning
a3. reduces to simple majority vote in 2-candidate case.
a4. neutrality.
a5. method is deterministic aside from tiebreaks which (if any) are random
a6. adding a new candidate to the election whom all voters unanimously rank unique-bottom, does not change the winner.

About replacements for a3. As we shall discuss below, due to Campbell-Kelly 2003 [5], criterion a3 can be replaced by demand b3 that no "vetoer" exists, combined with the a5 demand that the system be deterministic (chance is not employed except where required by symmetry, i.e. in the case of true ties; or, better, we can regard any system obeying a5 as being 100% deterministic but simply outputting tied-winner *sets*; this interpretation is also ok), and the demand it be based on rank-order ballots. And this replacement is desirable since you get a stronger theorem, i.e. depending on weaker axioms. But leaving a3 as is, is convenient for the purposes of the proof below.

The droppability of a5. Our Main Theorem drops a5 just by subsuming it into the definition of "nontrivial voting system." ("Trivial" voting systems that just pick a winner randomly, are excluded.) It is simplest, although not actually necessary for our proof to work, for the tiebreaking to give equally likely win probabilities to all co-winners. But it actually will suffice for us if the probabilities are fixed and positive for each co-winner in each scenario.

The droppability of a3. A3 is the axiom that the voting system in the 2-candidate case reduce to simple majority rule. It is known to be a consequence of determinism, neutrality, anonymity and "positive responsiveness"[4] and hence really is *not needed* per se[5]:

May's theorem on *2-candidate* elections [10][6, 16]: *A group decision function meets c1, c2, and c3 below if and only if it is the simple majority method:*

[4] Positive responsiveness: voting for X top breaks an X-containing perfect tie in favor of X, plus monotonicity

[5] We thank Forest Simmons for pointing that out.

c1. It is symmetric under permuting the voters. (anonymity)
c2. Reversing each preference reverses the group preference. (neutrality)
c3. If the group decision was 0 or 1 and a voter raises a vote from -1 to 0 or 1 or from 0 to 1, then the group decision is 1. (positive responsiveness)

Also, even better for our purposes, simple majority rule in the 2-candidate case is a consequence of AFB (actually strategyproofness, but this is the same as AFB in the 2-candidate rank-equality-forbidden case with an odd number of voters, and "odd" is the only parity we shall need) and non-dictatorship (which is implied by the nonexistence of a vetoer) and determinism. This is due to Campbell & Kelly 2003 [5], and note that they do not require anonymity axiom c1.

The droppability of a6. A6 can be dropped if we are using Campbell-Kelly 2003 to replace a3 with "no vetoer." That is because the proof of the Theorem shall only use a6 in a 3-candidate situation with an odd number of voters where removal of the always-ranked-last candidate yields 2-candidate simple majority vote; and those situations were *already* covered by the Campbell-Kelly 2003 theorem about (their somewhat more general notion of) "simple majority rule."

Further Remarks on a3: Both the May and Campbell-Kelly results above have versions that work even if equal rankings are allowed in ballots.

Also, majority-rule is a consequence of Neutrality, Anonymity, Pareto criterion (if all voters say $A > B$ then B cannot win), odd number of voters, and Independence of Irrelevant Alternatives. (For this result see Maskin [9] and it was improved by Campbell & Kelly [4].)

Two more characterizations of simple majority rule, both of which strike me as somewhat silly,[6] are [2][20].

Asan & Sanver [2] obtain simple majority rule from Neutrality, Anonymity, Pareto (if all voters say $A > B$ or $A = B$ with at least some saying $A > B$, then B cannot win), and a partition-consistency property (if both subdistricts say A wins or ties with at least one saying A wins, then A wins in the combined country).

ICC = Clone immunity for the purposes of this proof (and it also is common usage) is the demand that this be true:

> If clones of C are added to the election, that does not affect the winner (except perhaps up to replacement of the winner by a clone).

Here "clones" have to be contiguous in all rank-orders (for rank-order voting systems). There can be slight preferences among the clones e.g. some voter prefers $C_4 > C_1 > C_2 > C_3$, *but* these preferences are assumed to have *far smaller* strength than C versus a non-clone of C, e.g. far smaller strength than any comparisons like $D > C$ or $C > G$ or (for that matter) $A > B$. Therefore *range* voters will always vote clones almost-equal, to within ϵ say, where we will allow ourselves to take the limit $\epsilon \to 0$. That is the definition used in Tideman's book [15], and Tideman invented [18] the clone-immunity concept.

Under this definition range voting is clone-immune, and so are Schulze-beatpath voting [12][15], and IRV (instant runoff voting)[7]; but Plurality, Borda, Copeland[8], and Approval voting[9] are not clone-immune (and Tideman's book agrees with all these statements).

Mike Ossipoff's elegant wording: "A clone-set is a set of candidates between whom no one has voted any other candidate(s)."

AFB: the Favorite Betrayal Criterion:[10]

[6] In particular, the Asan-Sanver [2] result actually seems a trivial corollary of the far stronger Smith & Young [13][19] theorem we'll mention later, which had been proved over 20 years previously.

[7] IRV [17]: Votes are rank-orderings. The candidate top-ranked on the fewest ballots is eliminated (from the election, and from all ballots), reducing it to a $(C-1)$-candidate election, and the process continues until only one candidate remains – the winner.

[8] Copeland: Your vote is a rank-ordering of the candidates. A candidate X wins "pairwise" against some opponent Y if more voters say $X > Y$ than say $Y > X$. A candidate with the most pairwise victories wins.

[9] Approval [3]: Your vote is the set of candidates you "approve." Most-approved candidate wins.

[10] This has sometimes been called the "weak" Favorite Betrayal Criterion. We shall not need any stronger version.

The favorite betrayal criterion was proposed by Mike Ossipoff in web posts and both he and I consider it very important to make democracy work.

> AFB is the demand that it is never strategically forced for any voter to rank his true favorite, strictly below topmost. (A "strategically forced" move means, if you don't make that move, the election result comes out worse, in terms of expected utility, from your point of view.)

Really this all depends on the candidate-*utility numbers* for that voter, which is something we (somewhat badly) will not completely exhibit in the proof below. To fill in the gaps, one really should point out at various places how to construct explicit utility numbers to make it clear that voter-betrayal decisions were indeed strategically forced, otherwise the utility would be worse. We usually have not done this because dreaming up appropriate utility numbers for the voter in question, is usually a triviality, and actually giving numbers every time would have made the proof unbearable.

Range Voting obeys AFB and ICC:
Obviously, range-voting your favorite with any score below 9 lessens his chances to win, and leaves the relative winning chances of all the other candidates unaffected. I.e. it is just stupid. Hence range voting obeys AFB. Also obviously, if every range-voter scores a "clone" Q' of some candidate Q the same as Q to within $\pm\epsilon$, then in the limit $\epsilon \to 0+$ the same election result (up to replacement by a clone) will still occur. Hence range voting obeys ICC.

We should remark that, not surprisingly, the set of voting systems that suffer favorite betrayal (e.g. see table 1) is highly similar to the set of voting systems that lead over time to (≤ 2)-party dominated countries. In particular, Plurality voting and IRV both lead to 2-party domination ("Duverger's law,"[11] supported by a great deal of historical evidence).

About the proof's strategy:
The proof will work by demonstrating that, if a single-winner voting system based on pure-rank-order-ballots satisfying axioms a1-a6 did exist, then we could deduce a logical contradiction.

If any of these properties a1-a6 are violated in any election situation, then they are violated by that voting system, period; and we are done.

In the proof, a lot of little election situations will be considered, and if in any one of those situations, there is an ICC violation or an AFB violation (with some utility values and some way to alter some vote to betray), then game over and proof done.

We are allowed to use the assumption there is no useful way to betray a favorite, or use the assumption ICC is true, or any other axiom, to create new election situations and to deduce the winners in those new situations. (If those deductions were wrong, the proof would be complete because a contradiction would have been found.) Then in these new situations, we can aim to complete the proof by finding some other contradiction.

As a note on tactics, if we alter election-scenario 1 to get election 2 and make deductions about elections 1 and 2 using made-up utility values and reasoning about favorite-betrayal, then it is *not necessary* that the utility values in the two elections be consistent. That is because the voting system acts based *solely on the votes* and works in *ignorance* of honest utility values.

Proof of the Equivalent Theorem:
(We've already discussed why, by results of previous authors, the two theorems are equivalent.) The proof will work by demonstrating that, if a single-winner voting system based on pure-rank-order-ballots satisfying axioms 1-6 did exist, then we could deduce a logical contradiction. So suppose such a system existed. Consider these 3 votes:

$$A > B > C, \;\; C > A > B, \;\; B > C > A.$$

By symmetry axiom 4 this is a perfect 3-way tie with win probabilities 1/3, 1/3, 1/3. However we shall argue under axioms 1-3 that A must win, which is a contradiction that establishes the proof. [Really, we shall go through all the 7 possible winners ABC-tie, BC, AC, AB, and A, B, C proving none of these 7 possibilities are allowed

[11] http://RangeVoting.org/Duverger.html

by our axioms, which is the sought contradiction. However, we focus on the two cases ABC and A because they remain after the other 5 cases are eliminated, which proves "both are true" which is the requisite contradiction that establishes the proof.] So: If A does not win, then B or C does (or some sort of tie; we'll consider the cases below).

If B wins (or if AB tie), then the $C > A > B$ voter would betray C to vote $A > C > B$ getting

$$A > B > C, \;\; A > C > B, \;\; B > C > A$$

and then $\{B, C\}$ is a clone set and hence by axioms 2 and 3 then A must win and hence the betrayal worked and hence we get a contradiction with axiom 1. (In "slo-mo" that is: the votes are really $A > BC$, $A > BC$, $BC > A$ in a *two*-candidate election, which A wins by axiom 3, and when we clone BC into two candidates B and C, A still must win by axiom 2.) This betrayal would be a utility improvement from the point of view of that voter if her vote really is "$C > A >> B$", i.e. if her utility for B is greater than her average utility for $\{A, B, C\}$. To see that this betrayal was strategically forced, we also have to note that the alternate dishonest vote (which is not a C-betrayal) $C > B > A$, would not work since B still would win [here's how we know that: $A > B > C$, $C > B > A$, $B > C > A$; now $\{B, C\}$ is a clone set, so by axioms 2 and 3 we know B or C must win; but winner here must be B and not C (and not BC tie) because if it were C or BC tie then the $A > B >> C$-voter could betray: $B > C > A$ causing $B > C > A$, $C > B > A$, $B > C > A$ in which case B must win by axiom 6.]

If C wins, or if BC tie, or if ABC tie, or if AC tie, then the $A > B > C$-voter (whom for this purpose we assume feels $A > B >> C$) can betray A to vote $B > A > C$ getting $B > A > C$, $C > A > B$, $B > C > A$ whereupon $\{A, C\}$ is a clone set and hence by axioms 2 and 3 then B must win and hence the betrayal worked (assuming this voter had utilities such that B was valued above the mean utility of A,B,C) and hence we get a contradiction with axiom 1. [The alternate dishonest vote which is not an A-betrayal, $A > C > B$, would not work since C still would win with votes $A > C > B$, $C > A > B$, $B > C > A$ because $\{A, C\}$ is a winning clone set and if A wins (or AC tie) then $B > C >> A$ voter betrays: $C > A > B$ to make C win: $A > C > B$, $C > A > B$, $C > A > B$ by axiom 6.]

Q.E.D.

The "goodness" of the criteria: This theorem is not claiming that criteria a1-a6 or b1-b4 are good or bad (although they happen to all sound pretty good to me), and not claiming anybody necessarily should accept or reject them. It simply is claiming that it is logically unachievable to satisfy all of them at once by a rank-order voting system.

Range voting "achieves the impossible": But (normalized) range voting *does* satisfy all of them at once. [It is very nice when you can prove *every* voting system cannot do something – even voting systems nobody has ever invented yet.] We'll discuss that more precisely soon.

Remarks on other voting systems: *Antiplurality* voting[12] obeys all 6 axioms except for a2 (and a6). You can make a version of antiplurality voting that obeys a6 by making a last-place-vote count -1 and adding the tie-break rules that a second-to-last-ranking-vote counts $-\epsilon$, a third-last-vote counts $-\epsilon^2$, etc. in the limit $\epsilon \to 0+$ (and a candidate with greatest summed score wins).

I think there are also an infinite number of other rank-ballot systems avoiding favorite-betrayal, e.g. weighted positional systems depending on the last $C - 2$ rankings only, where C is the number of candidates.

Approval Voting obeys all 6 axioms except for a2 (provided we are sufficiently generous about a3 and a6; there is no doubt these are satisfied "in practice").

Random ballot obeys all the axioms, and so does *random pair*[13]. Both these systems were introduced by A.Gibbard. But we exclude them both as not deterministic enough, and similarly

[12] Antiplurality: Your vote is naming one candidate. The *least*-named candidate wins.

[13] Random pair: select two candidates at random, eliminate all the others, and then elect whichever pair-member would win a simple majority vote.

#voters	their vote
8	$B > C > A$
6	$C > A > B$
5	$A > B > C$

Table 1: FAVORITE BETRAYAL, or HOW DISHONEST EXAGGERATION CAN PAY. In this 19-voter example there is a Condorcet cycle, and the winner is B under numerous voting systems including Plurality, Borda, Schulze-Beatpaths, IRV, Copeland, Tideman Ranked Pairs, etc.
But if the 6 $C > A > B$ voters insincerely switch to $A > C > B$ ("betraying their favorite" C) then A becomes the winner under all these voting systems, which in their view is a better election result.

exclude trivial systems like random winner.

Schulze beatpaths voting obeys all 6 axioms except for a1. So does Instant Runoff Voting (at least if we are sufficiently generous about the random-tiebreaking axiom a5). Table 1 is a general-purpose example that shows AFB failure for these (and many other) voting systems. **Remark on the lack of necessity of ties in the argument:** Chris Benham suggests the proof might be less confusing (fewer worries about ties etc.) if we modify the initial example to replace the three voters with three identically voting equal-sized large factions and then adding one fickle bullet-voter.

#voters	their vote
33	$A > B > C$
33	$C > B > A$
33	$B > C > A$
1	? (A or B or C)

In this modified version of "Election 1," we could assume some perturbed symmetry axiom that the lone truncator must determine the winner. This should lead to a slightly different theorem statement with a somewhat simpler proof.

What if range votes are *not* necessarily normalized?

Range voting obeys all 6 axioms if all range votes are "normalized" (normalization can be assumed if voters are not total strategic idiots).

However, with possible-idiot voters, range fails axiom a3 (a fact does not bother me much...). Furthermore, with automatic renormalization after the votes are cast but before they are counted, range voting would violate axiom a6.[14]

These failures are slightly embarrassing if our goal is to find a set of criteria range voting obeys but rank-ballot methods fail. However, there are several ways one can use (and we shall use) to completely eliminate this embarrassment:

- FIX #1: We can either trust range voters not to be idiots in 2-candidate elections (to make axiom a3 hold), or we may, e.g. slightly *rephrase* axiom a6 as
  a6′. Adding a new candidate to the election whom all voters unanimously rank exactly ϵ below their previous bottommost does not change the winner in the limit $\epsilon \to 0+$.

- FIX #2 (which is the one we prefer and is the one used in our Main Result): we can simply *discard* axiom a3 by use of characterizations of simple majority rule as described previously (e.g. Campbell-Kelly [5]), and then we are free to use unnormalized range voting *without* any need to "trust" voters not to be stupid. We just let them be stupid if they want. This fix is excellent since discarding axioms is always a fine thing. Note that then we do not even need to assume anonymity in the Theorem, aside from (with Campbell-Kelly) a requirement of non-dictatorship.

Then we have indeed proven a sense in which range is superior to *every* pure-rank-ballot voting method, and using two of the most important voting criteria AFB and ICC.

3 What if rank-equalities are allowed in ballots?

I am presently unable to settle the question of whether AFB and ICC are incompatible if equalities are permitted.

How to tackle this kind of problem: In principle we can solve the existence problem by constructing a voting system satisfying both properties. Or, we could prove nonexistence

[14] Thanks to Markus Schulze for pointing that out.

purely mechanically, for some specific number C of candidates and V of voters (e.g. $C = 7$, $V = 15$) by simply examining every possible such election and every possible voting system with that C and V. Although this is a finite number of configurations, it appears to be well beyond the capabilities of the computers on this planet.

However, we now shall obtain a significant partial result.

Our approach: We can prove ICC is incompatible with an inequivalent version of AFB (which I do not preferentially endorse) call it AFB′:

> AFB′ = "Raising favorite to top rank must not decrease expected utility."

AFB′ is easier to work with than AFB, but I regard it as of less interest. AFB⟹AFB′, but the reverse implication does not hold.

Forest Simmons offers the following comparison: AFB says that voters can (without being strategically foolish) remain loyal to their favorite if they are clever enough. AFB′ says that remaining loyal to your favorite will *never* mess things up, even if you are *not* very clever. Simmons argues that the latter is what people want.

Theorem: *AFB′, ICC, neutrality, the assumption that in a perfect 3-way tie you break ties randomly with all tiers getting nonzero win probabilities, and finally, reduction to simple majority vote in the 2-candidate case (which assumption again can be replaced by simpler ones), are logically incompatible in any single-winner election method based on rank-order ballots with rank-equalities permitted.*

Proof: Begin with the election

#voters	their vote
3	$A = B > C$
3	$C = A > B$
3	$B = C > A$
2	$A > C > B$
2	$B > A > C$
2	$C > B > A$

which by symmetry this is a perfect 3-way tie. This election is quite useful. It had previously been used by Kevin Venzke[15] to demonstrate the fact that every Condorcet voting method (regardless whether rank-equalities are allowed in ballots, or forbidden) fails AFB.

Now suppose the three $A = B > C$ voters all betray their co-favorite B to get

#voters	their vote
3	$A > B > C$
3	$C = A > B$
3	$B = C > A$
2	$A > C > B$
2	$B > A > C$
2	$C > B > A$

and then the $B > A > C$ voters (regarded as $B > A >> C$), all betray their solo favorite B, to get

#voters	their vote
3	$A > B > C$
3	$A = C > B$
3	$B = C > A$
2	$A > C > B$
2	$A > B > C$
2	$C > B > A$

and finally the $A = C > B$ voters betray their co-favorite C to get

#voters	their vote
3	$A > B > C$ *
3	$A > C > B$ *
3	$B = C > A$
2	$A > C > B$
2	$A > B > C$ *
2	$C > B > A$

where the *s indicate dishonest votes. A wins in this scenario with 100% probability (by ICC and 2-candidate majority using the clone set $\{B, C\}$). So the net effect of these 3+2+3=8 betrayal decisions, was to cause a result all 8 of the betrayer-voters prefer. I.e, the betrayals worked.

Now consider these 8 betrayers changing their votes to the betrayal-vote one by one. At the start-end of the 8-betrayal chain we have an ABC tie. At the finish-end, A wins (a result all betrayer-types prefer). So at some point in

[15] See `http://RangeVoting.org/VenzkePf.html`.

the chain, there must have been a beneficial (to that betrayer) election-result-change.

This proves a single betrayal must work in some election situation. However... it remains possible that some non-betraying dishonest vote also works. But with the weakened definition AFB′ of AFB that is not an issue. **Q.E.D.**

4 Other property-sets range voting satisfies but no rank-order method can

There are many other property-sets that range voting satisfies whereas no rank-order method can simultaneously satisfy them. Ours is appealing because AFB and ICC seem very important properties to make democracy work. But other property sets have their appeal too.

Theorem. *These five properties are incompatible in a single-winner voting system based on pure-rank-order ballots:*
d1. partition-consistency. (That is, if X wins in district 1 and in district 2, then X must win in the combined 2-district country.)
d2. AFB
d3. "responsiveness at top" – raising a candidate from second-top to top in your vote (by swapping those two positions) in some election situations actually increases his winning chances.
d4. anonymity
d5. neutrality.

Proof sketch. The hard part was done by J.H. Smith [13] and G.P.Young [19] in the 1970s. These authors independently showed that any rank-ballot system obeying axioms d1, d4, and d5 had to be a "composition of weighted positional systems." But then that is readily seen to be incompatible with properties d2 and d3 (e.g. by considering a suitable cyclic-tie situation).
Q.E.D.

Theorem. *These four properties are incompatible in a single-winner voting system based on pure-rank-order ballots:*
e1. three-candidate semi-honesty. (That is, in a 3-candidate election, it is never strategically forced to dishonestly vote as though $X > Y$ when your honest view is $Y > X$.)
e2. determinism.
e3. no dictator.
e4. unanimously top-ranked candidate must win.

Proof sketch. This is an immediate consequence of the Gibbard-Satterthwaite strategyproofness theorem (see [14]c[16]).
Q.E.D.

Theorem. *These three properties are incompatible in a single-winner voting system based on pure-rank-order ballots:*
f1. no dictator.
f2. If every voter prefers A to B then so does the group. (This implicitly assumes the voting system outputs not only a winner but also a rank-order of all finishers. You can get a rank order even from a system without one by ranking the winner first, then delete that candidate from all ballots and ask who would have won then - rank him "second" – and so on.)
f3. "independence of irrelevant alternatives": The relative positions of A and B in the group ranking depend on their relative positions in the individual rankings, but do not depend on the individual rankings of any "irrelevant alternative," i.e. other candidate, C; to word it more precisely, we shall demand that if C is deleted from all ballots, then whether A finishes ahead of or behind B, is unaffected.

Proof sketch. This is an immediate consequence of K.Arrow's impossibility theorem (see [14]c[17]).
Q.E.D.

Theorem. *Every single-winner voting system based on rank-order ballots (with equalities either forbidden or permitted – both work) must suffer from at least one of the following paradoxes:*
p1. No-Show Paradox: A voter is better off (in the sense the election result improves from her point of view) by not voting than voting honestly.
p2. Condorcet failure: A "beats-all winner" candidate X (who defeats all opponents Y in elections based on the given votes but with all candidates besides X and Y removed) still loses the election.
p3. District partitioning paradox: A candidate

[16] Also discussed `http://RangeVoting.org/GibbSat.html`.
[17] Also discussed `http://RangeVoting.org/ArrowThm.html`.

wins in every district, but loses the general election.
p4. More-is-less Paradox (monotonicity failure): If the winner had been ranked higher by some voters, another candidate would have won.

Proof: This claim was made in [7]. But actually, just avoiding p1 and p2 *alone* is impossible, and that is true whether or not rank-equalities are permitted; a proof, due to Markus Schulze and really dating back to Herve Moulin, is found in [14]c.
Q.E.D.

5 Summary and Moral

In all of these impossibility theorems, Range Voting "accomplishes the impossible" by avoiding all the paradoxes p1-p4 and obeying all the criteria b1-b4, d1-d5, e1-e4, f1-f3. Of course, the reason that is possible is that the impossibility theorems all concern rank-order voting systems, and range voting isn't one. The moral of this paper – and also of my earlier [14]b – is that rank-order ballots are inherently a stupid voting system design idea, and the preoccupation of the political science community on them over the last 100 years for no good reason (e.g. range voting is not even *mentioned* in most political science books) has been a tremendous waste of time.

6 Acknowledgment

I thank Forest Simmons for helpful and/or inspirational comments. Indeed, a day or two after I produced my proof, Simmons produced his own (somewhat simpler) proof of a somewhat weaker result. You can see it at the web page corresponding to this paper.[18]

References

[1] Center for Range Voting `http://RangeVoting.org`

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a. #56 Range Voting;
b. #59 Voting schemes based on candidate-orderings or discrete choices regarded as harmful;
c. #79 The voting impossibilities of Arrow and of Gibbard & Satterthwaite;
d. #95 The 3-candidate left-middle-right scenario;
e. #96 Ants, Bees, and Computers agree Range Voting is best single-winner system;
f. #97 Completion of Gibbard-Satterthwaite impossibility theorem; range voting and voter honesty.

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