More "new" theorems in vector calculus

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Abstract — **Two well known theorems in 3D vector calculus are Gauss's divergence theorem (actually valid in n dimensions), and Stokes' theorem. I recently found a third natural theorem of that ilk. The present short note completes the cycle by finding the fourth and last one, plus finding the two more theorems that arise if scalar functions are admitted. All of the 3 "new" theorems are disguised forms of old ones.**

Keywords — Stokes, Gauss, divergence theorem.

1 The previous three theorems

In the below, let all functions, curves, and surfaces be sufficiently smooth[1], and assume all integrals are finite and exist. Assume a right-handed x, y, z coordinate system. For standard vector notation (e.g. $\vec{c} = \vec{a} \times \vec{b}$, $\ell = \vec{a} \cdot \vec{a} = |\vec{a}|^2$) meanings see [1][2]; $\vec{\nabla}$ denotes (in 3D) the differential operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1}$$

Gauss's divergence theorem states the equality of these two scalars:

$$\underbrace{\iint_V \cdots \int}_{n \text{ integrals}} \vec{\nabla} \cdot \vec{F} \, \mathrm{d}^n \vec{x} = \underbrace{\iint_{\partial V} \cdots \int}_{n-1 \text{ integrals}} \vec{F} \cdot \mathrm{d}\tilde{\mathrm{A}} \tag{2}$$

where V is some n-dimensional domain, ∂V is its $(n-1)$-dimensional bounding surface, $\mathrm{d}\vec{A}$ is the (outward pointing) vectorial element of surface $(n-1)$-area, and $\mathrm{d}^n\vec{x} = \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \ldots \mathrm{d}x_n$ is the vectorial element of n-volume. **Stokes' theorem** states the equality of these two scalars:

$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{A} = \int_{\partial D} \vec{F} \cdot \mathrm{d}\vec{\ell} \tag{3}$$

where D is a topological disk in 3-space (i.e., a region homeomorphic to

$$\{(x, y) \text{ such that } x^2 + y^2 \le 1\},$$

and ∂D is its bounding curve (homeomorphic to $\{(x, y) \text{ such that } x^2 + y^2 = 1\}$). The infinitesimal element of arc length pointing in the tangent direction to the curve (going clockwise as viewed looking in the $\mathrm{d}\vec{A}$ directions) is $\mathrm{d}\vec{\ell}$.

Both of these theorems have the form: a natural integral over a boundary equals the integral over the region itself. of some kind of derivative.

Generations of mathematicians have concluded that there are *two* natural kinds of "products" of 3-vectors, namely the scalar-valued dot product

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{4}$$

and the vector-valued cross product

$$\vec{A} \times \vec{B} = (a_2 b_3 - a_3 b_2, \, a_3 b_1 - a_1 b_3, \, a_1 b_2 - a_2 b_1). \tag{5}$$

(Consequently there are only three especially natural kinds of "derivatives" for 3-vectors, namely grad, div and curl: $\vec{\nabla} F$, $\vec{\nabla} \cdot \vec{F}$, and $\vec{\nabla} \times \vec{F}$, where note grad is generally applied to *scalar* F.) Furthermore, there are only *two* nontrivial natural kinds of regions and boundaries in 3-space, namely: 3D regions with 2D surfaces bounding them (as in Gauss's divergence theorem), and 2D surfaces with 1D curves bounding them (as in Stokes' theorem). This tells us that there are only 4 possible thinkable kinds of theorems of these general forms concerning vector-valued functions $\vec{F}$. If *scalar*-valued functions F are admitted, though, there then would be two additional possible kinds of theorems[2]

Just recently [3] I found the **third such theorem.** It states the equality of these two 3-vectors:

$$\iint_D (\vec{\nabla} \cdot \vec{F}) \mathrm{d}\vec{A} - (\vec{\nabla} \vec{F}) \mathrm{d}\vec{A} = \int_{\partial D} \vec{F} \times d\vec{\ell}. \tag{6}$$

Here $\vec{\nabla}\vec{F}$ denotes the 3×3 matrix whose i-down j-across entry is $\frac{\partial}{\partial x_i} F_j$. It multiplies the column-vector $\mathrm{d}\vec{A}$. EQ 6 has applications in electromagnetism. It may be regarded as a disguised form of Stokes' theorem and hence as "not new" – although the disguise is fairly heavy, and EQ 6 also is expressible in terms of certain natural *2-dimensional* differential operators [3], in which case it is even heavier.

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[1] It will suffice if all surfaces have piecewise continuous unit outward normal vector, all curves have piecewise continuous unit tangent vector, and all functions have continuous derivatives.

[2] Trying to go the other way by considering all natural integrals of "derivatives" of F over regions R, and then trying to devise equal integrals over ∂R, in general won't work because most derivatives will include information indepndent of any boundary integral. The cases where it does work are already covered here.

Our purpose in this paper is to finish the job by presenting the fourth natural theorem – and also the two additional theorems admitted by allowing scalar fields.

That turns out to be quite easy because the fourth and fifth theorems are really merely lightly disguised forms of Gauss's divergence theorem.

2 The fourth Theorem

The present paper's new **Fourth Theorem** states the equality of these two *3-vectors*:

$$-\iiint_V (\vec{\nabla} \times \vec{F}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \; = \; \iint_{\partial V} \vec{F} \times \mathrm{d}\vec{A}. \tag{7}$$

Proof. Consider the jth component of the right hand side of EQ 7 (got by taking its dot product with $\vec{e}_j$, the vector with a 1 in the jth coordinate and 0s elsewhere):

$$\iint_{\partial V} \vec{e}_j \cdot (\vec{F} \times \mathrm{d}\vec{A}). \tag{8}$$

Use the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ to see this is

$$= \iint_{\partial V} (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{A}. \tag{9}$$

Apply Gauss's divergence theorem to get

$$= \iiint_V \vec{\nabla} \cdot (\vec{e}_j \times \vec{F}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{10}$$

Now employing the vector identity ([1] 10.31#5)

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{11}$$

and taking advantage of the fact that $\vec{\nabla} \times \vec{e}_j = \vec{0}$ since $\vec{e}_j$ is a constant vector, this is

$$= -\iiint_V \vec{e}_j \cdot (\vec{\nabla} \times \vec{F}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{12}$$

proving the Theorem. **Q.E.D.**

Example: Let $\vec{F}(x, y, z) = (y, -x, k)$ where k is any constant. Let V be the radius-R height-$2h$ cylinder

$$|z| < h, \quad x^2 + y^2 \le R^2. \tag{13}$$

Then the surface integral over the *curved part* ($|z| < h$, $x^2 + y^2 = R^2$) of the cylinder is $4\pi h R^2 \vec{1}_z$. To see that, use the fact that the area of this surface is $4\pi h R$ and the fact that

$$(y, -x, k) \times \frac{(x, y, 0)}{R} = \frac{(-ky, \, kx, \, x^2 + y^2)}{R}, \tag{14}$$

which on our surface is $(-ky/R, \, kx/R, \, R)$. The first two coordinates integrate to 0 by axial symmetry; the last coordinate integrates to $4\pi h R^2$. The surface integrals over each of the two *flat endcaps* ($z = \pm h$, $x^2 + y^2 \le R^2$) of the cylinder are $\vec{0}$ by axial symmetry. The volume integral of $-\vec{\nabla} \times \vec{F} = (0, 0, 2)$ over the volume (which is $2\pi h R^2$) of the cylinder is $(0, 0, 4\pi h R^2)$. The theorem is confirmed: $4\pi h R^2 \vec{1}_z = (0, 0, 4\pi h R^2)$.

EQ 7 has **applications in electromagnetism.** E.g., it shows that the rate of change of the integrated (over a volume V) magnetic field $\vec{B}$ is the same thing as the surface integral $\iint_{\partial V} \vec{E} \times \mathrm{d}\vec{A}$, where $\vec{E}$ is the electric field, and also the same thing as $\frac{-\partial}{\partial t} \iint_{\partial V} \vec{P} \times \mathrm{d}\vec{A}$, where $\vec{P}$ is the vector potential ($\vec{\nabla} \times \vec{P} = \vec{B}$) and t is time.

3 The fifth Theorem

The **Fifth Theorem** states the equality of these two n-*vectors*:

$$\underbrace{\iint_V \cdots \int}_{n \text{ integrals}} \vec{\nabla} F \, \mathrm{d}^n \vec{x} \; = \; \underbrace{\iint_{\partial V} \cdots \int}_{n-1 \text{ integrals}} F \mathrm{d}\vec{A}. \tag{15}$$

Proof. Consider the jth component of the right hand side of EQ 15 (got by taking its dot product with $\vec{e}_j$):

$$\iint_{\partial V} \vec{e}_j \cdot (F \mathrm{d}\vec{A}). \tag{16}$$

By Gauss's divergence theorem this is

$$= \iiint_V \frac{\partial F}{\partial x_j} \, \mathrm{d}^n \vec{x}. \tag{17}$$

Q.E.D.

If V is an axis-aligned hypercube, this fifth theorem is just the Fundamental Theorem of Calculus (FToC). Thus EQ 15 is a pleasant n-dimensional generalization of the FToC. For an **electrostatic application**, let F be the scalar potential ($\vec{\nabla} F = \vec{E}$).

Example: Let $n = 2$ and let $F(x, y) = x$. Let V be the unit disc $x^2 + y^2 \le 1$. Then $\oint F \mathrm{d}\vec{\ell}$ around the unit circle is

$$\int_0^{2\pi} \cos\theta \, (\cos\theta, -\sin\theta) \, \mathrm{d}\theta = (\pi, 0). \tag{18}$$

Meanwhile double-integrating $\vec{\nabla} F = (1, 0)$ over the unit disc (of area π) yields $(\pi, 0)$. The theorem is confirmed.

4 The sixth Theorem

The **Sixth Theorem** states the equality of these two 3-*vectors*:

$$\iint_D \begin{pmatrix} 0 & \frac{\partial F}{\partial z} & \frac{-\partial F}{\partial y} \\ \frac{-\partial F}{\partial z} & 0 & \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} & \frac{-\partial F}{\partial x} & 0 \end{pmatrix} \mathrm{d}\vec{A} \; = \; \int_{\partial D} F d\vec{\ell}. \tag{19}$$

This is merely Stokes' theorem repeated three times, the first Stokes case

$$\iint_D [\vec{\nabla} \times (F, 0, 0)] \cdot \mathrm{d}\vec{A} \; = \; \int_{\partial D} (F, 0, 0) \cdot d\vec{\ell} \tag{20}$$

accounting for the first row of the matrix, and the others are analogous.

References

[1] I.S. Gradshteyn & I.M. Ryzhik: Table of Integrals, Series, and Products, Academic Press. Chapter 10 is on vector identities and vector calculus.

[2] J.D.Jackson: Classical electromagnetism, Addison-Wesley 3rd edition 1998.

[3] Warren D. Smith: A new theorem in vector calculus, `http://math.temple.edu/~wds/homepage/newvec.{dvi,ps,pdf}`