New theorems in vector calculus

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Abstract — **Two well known theorems in 3D vector calculus are Gauss's divergence theorem (actually valid in n dimensions), and Stokes' theorem. We find new ones. They have interesting consequences in elementary classical electromagnetism. There is a natural way to classify possible theorems of this kind and we have found every theorem the classification admits. These theorems ought to be in the usual undergraduate vector calculus and electromagnetism textbooks, but aren't.**

Keywords — Stokes, Gauss, divergence theorem, Lorentz force, current loop.

1 Recapitulation of the theorems of Gauss and Stokes

In the following, let all functions, curves, and surfaces be sufficiently smooth,[1] and assume all integrals are finite and exist. Assume a right-handed x, y, z coordinate system. For standard vector notation (e.g. $\vec{c} = \vec{a} \times \vec{b}$ and $\ell = \vec{a} \cdot \vec{a} = |\vec{a}|^2$) meanings see [1][2]; $\vec{\nabla}$ denotes (in 3D) the differential operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1}$$

Gauss's divergence theorem states the equality of these two scalars:

$$\underbrace{\iint_V \cdots \int}_{n \text{ integrals}} \vec{\nabla} \cdot \vec{F} \, \mathrm{d}^n\vec{x} = \underbrace{\iint_{\partial V} \cdots \int}_{n-1 \text{ integrals}} \vec{F} \cdot \mathrm{d}\tilde{\mathrm{A}} \tag{2}$$

where V is some n-dimensional domain, ∂V is its $(n-1)$-dimensional bounding surface, $\mathrm{d}\vec{A}$ is the (outward pointing) vectorial element of surface $(n-1)$-area, and $\mathrm{d}^n\vec{x} = \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \ldots \mathrm{d}x_n$ is the vectorial element of n-volume. **Stokes' theorem** states the equality of these two scalars:

$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{A} = \int_{\partial D} \vec{F} \cdot \mathrm{d}\vec{\ell} \tag{3}$$

where D is a topological disk in 3-space (i.e., a region diffeomorphic to

$$\{(x, y) \text{ such that } x^2 + y^2 \leq 1\},$$

and ∂D is its bounding curve (homeomorphic to $\{(x, y) \text{ such that } x^2 + y^2 = 1\}$). The infinitesimal element of arc length pointing in the tangent direction to the curve (going clockwise as viewed looking in the $\mathrm{d}\vec{A}$ directions) is $\mathrm{d}\vec{\ell}$.

Both of these theorems have the form: a natural integral over a boundary equals the integral, over the region itself, of some kind of derivative.

2 Classification of possible theorems

Generations of mathematicians have concluded that there are *two* natural kinds of "products" of 3-vectors, namely the scalar-valued dot product

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{4}$$

and the vector-valued cross product

$$\vec{A} \times \vec{B} = (a_2 b_3 - a_3 b_2, \, a_3 b_1 - a_1 b_3, \, a_1 b_2 - a_2 b_1). \tag{5}$$

Consequently there are only three especially natural kinds of "derivatives" for 3-vectors, namely grad, div and curl: $\vec{\nabla} F$, $\vec{\nabla} \cdot \vec{F}$, and $\vec{\nabla} \times \vec{F}$, where note grad is generally applied to *scalar* F.

Furthermore, there are only *two* nontrivial natural kinds of regions and boundaries in 3-space, namely: 3D regions with 2D surfaces bounding them (as in Gauss's divergence theorem), and 2D surfaces with 1D curves bounding them (as in Stokes' theorem). This tells us that there are only $4 = 2 \times 2$ possible thinkable kinds of theorems of this general form concerning vector-valued functions $\vec{F}$.

If *scalar*-valued functions F are admitted, though, there then would be $2 = 2 \times 1$ additional possible kinds of theorems.[2]

3 The Third Theorem and its proof

The Third Theorem: These two *3-vectors* are equal:

$$\vec{B} - \vec{C} = \int_{\partial D} \vec{F} \times \mathrm{d}\vec{\ell} \tag{6}$$

[1] It will suffice if all surfaces have piecewise continuous unit outward normal vector, all curves have piecewise continuous unit tangent vector, and all functions have continuous derivatives.

[2] Trying to go the other way by considering all natural integrals of "derivatives" of F over regions R, and then trying to devise equal integrals over ∂R, in general won't work because most derivatives will include information independent of any boundary integral. The cases where it does work are already covered here.

where

$$\vec{B} = \iint_D \mathrm{TDD}(\vec{F})\,\mathrm{d}\vec{A} \tag{7}$$

and

$$\vec{C} = \iint_D \overrightarrow{\mathrm{TDG}}(\vec{F})\,\mathrm{d}A. \tag{8}$$

Here $\mathrm{TDD}(\vec{F})$ is the *t*wo-*d*imensional *d*ivergence of the projected version of $\vec{F}$ (projected down into the 2D tangent plane to the surface D) at the present point in 3-space. Finally $\overrightarrow{\mathrm{TDG}}(\vec{F})$ is the *t*wo-*d*imensional *g*radient (as a 3-vector) of the normal component of $\vec{F}$ (normal to the tangent plane to the surface D, and with the gradient taken in that tangent plane at the current point [of tangency]). Just as in Stokes' theroem, D is a topological disk in 3-space and ∂D is its bounding curve (going clockwise as seen looking along the directions of the normals $\vec{a}$ to the surface D). [See also EQs 9 and 10 below and the slick reformulation in EQ 18.]

Forward pointer: This therem will be reformulated, and re-proven, in §5. That other discussion is cleaner and the reformulation is better for some, but worse for other, purposes.

Proof sketch: Let $\vec{x} = (x, y, z)$. It suffices to prove it for *linear* functions $\vec{F}(\vec{x})$ only. Also it suffices to prove it merely in the case when D is a *triangle* in 3-space. [The rest will then follow by using the fact that all smooth functions are locally linear; subdividing our arbitrary smooth topological disk into tiny triangles, proof it is OK (in limit of tinyness) to neglect quadratic terms arising from non-flatness of the triangles, non-straightness of the triangle edges, and non-linearity of $\vec{F}$; and cancellation of the 1D integrals on interior triangle edges going both ways to get a $-$ sign cancelling a $+$ sign due to the bilinearity of the vector cross product $\times$ operation.]

Further, due to linearity of all three integrals with respect to $\vec{F}$, it suffices if we prove it only for a suitable set of *basis* functions $\vec{F}$. There are 12 obvious basis functions for the arbitrary linear functions mapping 3-vectors to 3-vectors, namely $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, $(x, 0, 0)$, $(y, 0, 0)$,..., $(0, 0, z)$. Finally, by rotational invariance, it suffices if our triangle lies *in a plane parallel to the xy plane.* We now proceed to the details.

(1) If $\vec{F}$ is any *constant* vector the statement is obviously just $0 = 0$ due to the fact the vector sides of a triangle sum to $\vec{0}$ since the triangle is a closed curve.

From now on by considering adding a constant offset in the z direction we may assume without loss of generality that our triangle lies in the xy plane itself, not just some parallel translate of it.

(2) If $\vec{F} = (0, 0, x)$ then the statement comes down to $-\int_D (1, 0, 0)dA = \int_{\partial D}(-x\mathrm{d}y, x\mathrm{d}x, 0) = (-A, 0, 0)$ where A is the area of our triangle D (which lies in the xy plane). Similarly if $\vec{F} = (0, 0, y)$ and the triangle lies in the xy plane then the statement comes down to $-\int(0, 1, 0) \times \mathrm{d}\vec{A} = \int_{\partial D}(-y\mathrm{d}y, y\mathrm{d}x, 0) = (0, -A, 0)$.

(3) If $\vec{F} = (0, 0, z)$ then the statement comes down to $0 = 0$ (the integrals of $z\mathrm{d}x$ and of $z\mathrm{d}y$ round a closed curve in an xy-parallel plane are both 0).

(4) If $\vec{F} = (z, 0, 0)$ and the triangle lies in the xy plane then the statement comes down to $0 = 0$. The right hand 0 is since $\int_{\partial D}(0, -z\mathrm{d}z, z\mathrm{d}y) = (0, 0, 0)$ if ∂D is a closed curve bounding a topological disk in the xy plane.

(5) If $\vec{F} = (y, 0, 0)$ then the statement comes down to $(0, 0, 0) = \int_{\partial D}(0, -y\mathrm{d}z, y\mathrm{d}y) = (0, 0, 0)$ since $\mathrm{d}z = 0$ and integrating $y\mathrm{d}y$ leads to $\pm$ cancellation.

(6) If $\vec{F} = (x, 0, 0)$ then the statement comes down to $(0, 0, A) = \int_{\partial D}(0, -x\mathrm{d}z, x\mathrm{d}y)$ and since $\mathrm{d}z = 0$ the first two coordinates are both 0. The integral of $x\mathrm{d}y$ is the area by a slabs argument (slabs $\mathrm{d}y$ wide and $x_2 - x_1$ in width). These 6 cases are the only ones that arise (any others are equivalent) so, **Q.E.D.**

Clearer formulation: How can the Third Theorem be formulated purely as math (instead of using English words in the description of the integrals $\vec{B}$ and $\vec{C}$)? Here's one way: Let the surface of a topological disk D be parameterized $(x, y, z) = \vec{W}(p, q)$. Let $\vec{t} = \partial\vec{W}/\partial p$ and $\vec{u} = \partial\vec{W}/\partial q$. Let $\vec{a} = \pm\vec{t} \times \vec{u}$ be a normal 3-vector to the surface D in 3-space so that $d\vec{A} = \vec{a}\mathrm{d}p\mathrm{d}q$. is the infinitesimal element of surface area. (The sign is chosen to make $\vec{a}$ have the correct orientation.) Then the first integral in the Theorem is

$$\vec{B} = \iint_D \left[\vec{t} \cdot \frac{\partial\vec{F}}{\partial p}|\vec{t}|^{-2} + \vec{u} \cdot \frac{\partial\vec{F}}{\partial q}|\vec{u}|^{-2}\right]\mathrm{d}\vec{A}. \tag{9}$$

The second integral is

$$\vec{C} = \iint_D \left(\vec{a} \cdot \frac{\partial\vec{F}}{\partial p}|\vec{t}|^{-2}\vec{t} + \vec{a} \cdot \frac{\partial\vec{F}}{\partial q}|\vec{u}|^{-2}\vec{u}\right)\mathrm{d}p\mathrm{d}q. \tag{10}$$

Remark: Stokes' and Gauss's theorems may be proven in much the same manner "subdivide into triangles and consider a basis set of linear functions" manner as our new Theorem (only proving them is easier).

4 Confirmatory Examples

Example #1. Let the surface D be the disk $x^2+y^2 < 1$, $z = 0$ and its bounding curve ∂D be the unit circle $x^2 + y^2 = 1$, $z = 0$. Let $\vec{F} = (x^2yz^2,\ xyz+3z+9,\ 5x+y^2z+7)$. Then we have

$$\vec{F} \times (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = ([(xy + 3)z + 9]\mathrm{d}z - (5x + 7 + y^2z)\mathrm{d}y, \tag{11}$$
$$(5x + 7 + y^2z)\mathrm{d}x - x^2yz^2\mathrm{d}z,\ x^2yz^2\mathrm{d}y - [(xy + 3)z + 9]\mathrm{d}x)$$

The integral of this around the unit circle (which is $\int_{\partial D} \vec{F} \times \mathrm{d}\vec{\ell}$) is

$$(0 - 5\pi, 0 - 0, 0 - 0) \tag{12}$$

since everything cancels out (by symmetry or $\mathrm{d}z = 0$) except for $\int_{\partial D} -5x\mathrm{d}y = -5\mathrm{area}(D) = -5\pi$. Meanwhile the first surface integral (using as parameters p, q just $p = x$ and $q = y$) is

$$\vec{B} = \iint_D [2xyz^2 + xz]\,(0, 0, 1)\,\mathrm{d}x\mathrm{d}y \tag{13}$$

which is $(0,0,0)$ by odd symmetry. The second surface integral is

$$\vec{C} = \iint_D (5, 2yz, 0)\,\mathrm{d}x\mathrm{d}y = (5\pi, 0, 0). \qquad (14)$$

Result: $-(5\pi,0,0) = (0-5\pi, 0-0, 0-0)$. The Theorem worked.

Example #2. Let $\vec{F} = (x,y,z)$, let the curve be the unit circle $x^2+y^2=1$, $z=0$, and let the surface D be the hemisphere $x^2+y^2+z^2=1$, $z>0$. Then the curve integral is $2\pi\vec{1}_z$. The surface integrals are $\vec{B} = \iint 2\mathrm{d}\vec{A} = 2\cdot 2\pi\cdot\frac{1}{2}\vec{1}_z = 2\pi\vec{1}_z$ and $\vec{C} = \iint \vec{0}\mathrm{d}A = \vec{0}$ respectively. (In computing $\vec{B}$ we have used the fact that the TDD of $\vec{F}$ is 2, as opposed to $\vec{\nabla}\cdot\vec{F} = 3$, we have used the fact the surface area of the hemisphere is 2π, and we have used the fact (due to Archimedes' correspondence between the surface area of a sphere and the cylinder enclosing it) that the average height of the surface of a hemisphere is half its radius. $\vec{C} = \vec{0}$ is since the integrand is everywhere 0 since $\vec{F}(\vec{x})$ is normal to the sphere surface and of constant length on it.) The Theorem worked: $\vec{0} = \vec{0}$.

Example #3. Let $\vec{F} = (0,0,1)$, let the curve be the unit circle $x^2+y^2=1$, $z=0$, and let the surface be the hemisphere $x^2+y^2+z^2=1$, $z>0$. Then the curve integral is $\vec{0}$ by symmetry. The surface integrals are $\vec{B} = \iint 0\mathrm{d}\vec{A} = \vec{0}$ (since the TDD of a constant vector is 0) and $\vec{C} = \iint \vec{0}\mathrm{d}A = \vec{0}$ (since the gradient of a constant vector is 0) respectively, proving once again that $\vec{0} = \vec{0}$.

Example #3 reveals a subtlety: The 2D divergence of $\vec{F}$'s projection into the tangent plane to our surface (TDD($\vec{F}$)), is generally *not* the same as the 2D divergence of $\vec{F}$'s projection onto the surface itself. Similarly, the 2D gradient of the normal-to-plane component of $\vec{F}$ (within that plane, i.e. TDG($\vec{F}$)) is generally *not* the same as the 2D gradient of $\vec{F}$'s normal component to the surface, on that surface. (The former, plane-based quantities are the ones my Theorem wants; the latter surface-based quantities are not. In example #3 the former both are $\vec{0}$, but the latter are both nonzero.)

Example #4. The computer-algebra package `MAPLE` has confirmed the Theorem for a *fully general* nonhomogeneous-quadratic polynomial map $\vec{F}(x,y,z)$ from $\mathbf{R}^3 \to \mathbf{R}^3$ in two cases:

1. where D is the hemisphere $x^2+y^2+z^2=1$, $z\geq 0$,
2. where D is the flat disk $x^2+y^2\leq 1$, $z=0$.

In both cases ∂D is the unit circle $x^2+y^2=1$, $z=0$. Also, I tried adding in some (not fully general) *cubic* terms, and the Theorem still passed the resulting tests. In these cases all the integrals are of trigonometric polynomials (if we employ spherical, or polar, coordinates, respectively) hence expressible in closed form.

The full details are too messy to include here, but the `MAPLE` scripts that confirm this are available electronically[3] and we'll now sketch how it goes for the hemisphere-quadratic. First we define

$$\vec{F}(\vec{x}) = \qquad (15)$$
$$(c^{(1)}_{11}x^2 + c^{(1)}_{12}xy + c^{(1)}_{13}xz + c^{(1)}_{22}y^2 + \cdots + c^{(1)}_{3}z + c^{(1)},$$
$$c^{(2)}_{11}x^2 + c^{(2)}_{12}xy + \cdots + c^{(2)},\ c^{(3)}_{11}x^2 + c^{(3)}_{12}xy + \cdots + c^{(3)})$$

The curve integral $\vec{I} = \int_{\partial D}\vec{F}\times\mathrm{d}\vec{x}$ may be done by computing $\vec{F}\times\mathrm{d}\vec{x}$ and then making the substitutions $x=\cos\theta$, $y=\sin\theta$, $z=0$, $\mathrm{d}x = -\sin\theta\mathrm{d}\theta$, $\mathrm{d}y = \cos\theta\mathrm{d}\theta$, $\mathrm{d}z=0$ and integrating from $\theta=0$ to 2π. The parameterized hemisphere is $\vec{x} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ for $0\leq\theta\leq 2\pi$ and $0\leq\phi\leq\pi/2$. We then compute $\vec{v} = \frac{\partial}{\partial\theta}\vec{x}$, $\vec{w} = \frac{\partial}{\partial\phi}\vec{x}$, $\vec{a} = \vec{w}\times\vec{v}$. We now compute the two surface integrals (using EQs 9 and 10 where we are here using ϕ and θ as the parameters, not p and q):

$$\vec{B} = \int_0^{\pi/2}\int_0^{2\pi}\left[\vec{v}\cdot\frac{\partial\vec{F}}{\partial\phi}|\vec{v}|^{-2} + \vec{w}\cdot\frac{\partial\vec{F}}{\partial\theta}|\vec{w}|^{-2}\right]\vec{a}\mathrm{d}\theta\mathrm{d}\phi \qquad (16)$$

$$\vec{C} = \int_0^{\pi/2}\int_0^{2\pi}\left(\vec{a}\cdot\frac{\partial\vec{F}}{\partial\phi}|\vec{v}|^{-2}\vec{v} + \vec{a}\cdot\frac{\partial\vec{F}}{\partial\theta}|\vec{w}|^{-2}\vec{w}\right)\mathrm{d}\theta\mathrm{d}\phi. \qquad (17)$$

Finally, we confirm that $\vec{I} = \vec{B} - \vec{C}$.

5 3D-only reformulation of Third Theorem

Adding $\vec{Q}$ to both $\vec{B}$ and $\vec{C}$ leaves the difference $\vec{B}-\vec{C}$ unaltered. Choose $\vec{Q}$ to be the directional derivative of the component of $\vec{F}$ normal to the tangent plane to D (in the normal direction to that plane) to get this
Slick Reformulation of the Third Theorem:

$$\boxed{\iint_D(\vec{\nabla}\cdot\vec{F})\mathrm{d}\vec{A} - (\vec{\nabla}\vec{F})\mathrm{d}\vec{A} \;=\; \int_{\partial D}\vec{F}\times d\vec{\ell}.} \qquad (18)$$

Here $\vec{\nabla}\vec{F}$ means the 3×3 matrix whose i-down j-across entry is $\frac{\partial}{\partial x_i}F_j$. (This is the transpose of F's Jacobian.) It multiplies the column-vector $\mathrm{d}\vec{A}$.

EQ 18 has the advantage that it is formulated purely in terms of the usual 3D differential operator $\vec{\nabla}$ rather than our invented 2D-inside-3D operators TDD and $\overrightarrow{\mathrm{TDG}}$. [On the other hand, in some applications, the original 2D-in-3D formulation might be more advantageous. `MAPLE` has also tested example #4 (extended with cubic terms) for the reformulated theorem.]

6 How to view it as Stokes in disguise

At first I suspected the Third Theorem was only a tiny consequence of an ultra-general theorem of Poincare [3] about differential forms on manifolds. Poincare's theorem subsumes both Stokes' theorem and the divergence theorem on n-manifolds as special cases. But that cannot be directly true because (on our 2-manifold D) it involves 3-vectors rather than 2-vectors; and any differential form on an n-manifold has some power of n components, but

[3] `http://math.temple.edu/~wds/homepage/works.html`

3 is not a power of 2. By that reasoning the Third Theorem really is "new."

But the slick reformulation in §5 suggested that the new Theorem is really just Stokes' theorem used 3 times with results linearly combined, with various altered functions employed inside the different Stokes invocations. This turns out indeed to be true; in a conversation with Yury Grabovsky (Temple Univ. Math. dept.) we were able to produce such a **second proof:**

Consider the jth component of EQ 18, i.e. (letting $\vec{e}_j$ denote the unit vector in the x_j direction)

$$\int_{\partial D} \vec{e}_j \cdot (\vec{F} \times \mathrm{d}\vec{\ell}). \tag{19}$$

Using the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ this is

$$= \int_{\partial D} (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{\ell}. \tag{20}$$

Applying Stokes' theorem this is

$$= \iint_D \vec{\nabla} \times (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{A}. \tag{21}$$

Now employing the vector identity ([1] 10.31#7)

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} \tag{22}$$

and taking advantage of the facts that $\vec{\nabla} \cdot \vec{e}_j = 0$ and $\vec{\nabla} \times \vec{e}_j = \vec{0}$ since $\vec{e}_j$ is a constant vector, this is

$$= \iint_D \left[\vec{e}_j(\vec{\nabla} \cdot \vec{F}) - (\vec{e}_j \cdot \vec{\nabla})\vec{F} \right] \cdot \mathrm{d}\vec{A} \tag{23}$$

$$= \vec{e}_j \cdot \iint_D (\vec{\nabla} \cdot \vec{F})\mathrm{d}\vec{A} - (\vec{\nabla}\vec{F})\mathrm{d}\vec{A} \tag{24}$$

Q.E.D.

So by *this* reasoning, the new theorem really is *not* "new:" it is merely a disguised form of Stokes' theorem. However, the disguise is fairly heavy, and in the original form involving *2-dimensional* differential operators it is even heavier.

I think the question of whether this all qualifies as "new" is subjective.[4] But I am quite confident that these theorems both ought to be mentioned in the usual undergraduate vector calculus textbooks, and aren't, so in that sense, it is definitely new.

We can now continue by finding additional theorems. That turns out to be quite easy because the fourth and fifth theorems are quite lightly disguised forms of Gauss's divergence theorem.

[4] Of course all results in real analysis depend on the same set of Axioms of real numbers, hence are not independent except in the extremely rare cases that they depend on disjoint axiom subsets. So the question of "newness" is always subjective.

7 The Fourth Theorem

The Fourth Theorem: The following two *3-vectors* are equal:

$$-\iiint_V (\vec{\nabla} \times \vec{F})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \;=\; \iint_{\partial V} \vec{F} \times \mathrm{d}\vec{A}. \tag{25}$$

Proof. Consider the jth component of the right hand side of EQ 25 (got by taking its dot product with $\vec{e}_j$, the vector with a 1 in the jth coordinate and 0s elsewhere):

$$\iint_{\partial V} \vec{e}_j \cdot (\vec{F} \times \mathrm{d}\vec{A}). \tag{26}$$

Use the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ to see this is

$$= \iint_{\partial V} (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{A}. \tag{27}$$

Apply Gauss's divergence theorem to get

$$= \iiint_V \vec{\nabla} \cdot (\vec{e}_j \times \vec{F})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{28}$$

Now employing the vector identity ([1] 10.31#5)

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{29}$$

and taking advantage of the fact that $\vec{\nabla} \times \vec{e}_j = \vec{0}$ since $\vec{e}_j$ is a constant vector, this is

$$= -\iiint_V \vec{e}_j \cdot (\vec{\nabla} \times \vec{F})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{30}$$

Q.E.D.

Example: Let $\vec{F}(x, y, z) = (y, -x, k)$ where k is any constant. Let V be the radius-R height-$2h$ cylinder

$$|z| < h, \quad x^2 + y^2 \le R^2. \tag{31}$$

Then the surface integral over the *curved part* ($|z| < h$, $x^2 + y^2 = R^2$) of the cylinder is $4\pi h R^2 \vec{1}_z$. To see that, use the fact that the area of this surface is $4\pi h R$ and the fact that

$$(y, -x, k) \times \frac{(x, y, 0)}{R} = \frac{(-ky,\, kx,\, x^2 + y^2)}{R}, \tag{32}$$

which on our surface is $(-ky/R,\, kx/R,\, R)$. The first two coordinates integrate to 0 by axial symmetry; the last coordinate integrates to $4\pi h R^2$. The surface integrals over each of the two *flat endcaps* ($z = \pm h$, $x^2 + y^2 \le R^2$) of the cylinder are $\vec{0}$ by axial symmetry. The volume integral of $-\vec{\nabla} \times \vec{F} = (0, 0, 2)$ over the volume (which is $2\pi h R^2$) of the cylinder is $(0, 0, 4\pi h R^2)$. The theorem is confirmed: $4\pi h R^2 \vec{1}_z = (0, 0, 4\pi h R^2)$.

8 The fifth Theorem

The Fifth Theorem: These two *n-vectors* are equal:

$$\underbrace{\iint_V \cdots \int}_{n \text{ integrals}} \vec{\nabla} F\, \mathrm{d}^n \vec{x} \;=\; \underbrace{\iint_{\partial V} \cdots \int}_{n-1 \text{ integrals}} F \mathrm{d}\vec{A}. \tag{33}$$

Proof. Consider the jth component of the right hand side of EQ 33 (got by taking its dot product with $\vec{e}_j$):

$$\iint_{\partial V} \vec{e}_j \cdot (F \mathrm{d}\vec{A}). \tag{34}$$

By Gauss's divergence theorem this is

$$= \iiint_V \frac{\partial F}{\partial x_j} \mathrm{d}^n \vec{x}. \tag{35}$$

Q.E.D.

If V is an axis-aligned hypercube, this fifth theorem is just the Fundamental Theorem of Calculus (FToC). Thus EQ 33 is a pleasant n-dimensional generalization of the FToC.

Example: Let $n = 2$ and let $F(x, y) = x$. Let V be the unit disc $x^2 + y^2 \leq 1$. Then $\oint F \mathrm{d}\vec{\ell}$ around the unit circle is

$$\int_0^{2\pi} \cos\theta \, (\cos\theta, -\sin\theta) \, \mathrm{d}\theta = (\pi, 0). \tag{36}$$

Meanwhile double-integrating $\vec{\nabla} F = (1, 0)$ over the unit disc (of area π) yields $(\pi, 0)$. The theorem is confirmed.

9 The sixth Theorem

The Sixth Theorem: These two 3-*vectors* are equal:

$$-\iint_D (\vec{\nabla} F) \times \mathrm{d}\vec{A} \;=\; \int_{\partial D} F d\vec{\ell}. \tag{37}$$

Proof. Upon writing

$$-(\vec{\nabla} F) \times \mathrm{d}\vec{A} \;=\; \begin{pmatrix} 0 & \frac{\partial F}{\partial z} & \frac{-\partial F}{\partial y} \\ \frac{-\partial F}{\partial z} & 0 & \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} & \frac{-\partial F}{\partial x} & 0 \end{pmatrix} \mathrm{d}\vec{A} \tag{38}$$

we see that EQ 37 is merely Stokes' theorem repeated three times, the first Stokes case

$$\iint_D [\vec{\nabla} \times (F, 0, 0)] \cdot \mathrm{d}\vec{A} \;=\; \int_{\partial D} (F, 0, 0) \cdot d\vec{\ell} \tag{39}$$

accounting for the first row of the matrix, and the others are analogous.

10 Electromagnetic Consequences

The right hand side of the Third Theorem is of course a very natural vector quantity, which arises in electromagnetism when computing the **Lorentz force** exerted by a magnetic field $\vec{F}(\vec{x})$ on a loop of wire ∂D circulating an electric current.

One immediate **corollary** of our Theorem's 3D-only reformulation in §5, is the following. Suppose a magnetic field $\vec{F}(\vec{x})$ obeys the Maxwell equation

$$\vec{\nabla} \cdot \vec{F} = 0 \quad \text{(no magnetic monopoles)} \tag{40}$$

Suppose D is a 2-dimensional topological disk surface in $\mathbf{R}^3$ which is such that, at all points $\vec{x} \in D$, the component of $\vec{F}(\vec{x})$ normal to D is *constant*. Then the Lorentz force on the current loop ∂D is zero: $\int_{\partial D} \vec{F} \times \mathrm{d}\vec{\ell} = \vec{0}$.

The Fourth Theorem also has application to electromagnetism. EQ 25 combined with the Maxwell equation $\vec{\nabla} \times \vec{E} = \frac{-\partial}{\partial t} \vec{B}$ shows that the rate of change of the integrated (over a volume V) magnetic field $\vec{B}$ is the same thing as the surface integral $\iint_{\partial V} \vec{E} \times \mathrm{d}\vec{A}$, where $\vec{E}$ is the electric field, and also the same thing as $\frac{-\partial}{\partial t} \iint_{\partial V} \vec{P} \times \mathrm{d}\vec{A}$, where $\vec{P}$ is the vector potential ($\vec{\nabla} \times \vec{P} = \vec{B}$) and t is time.

Finally, here is an electrostatic application of the Fifth Theorem. Let F be the scalar potential ($\vec{\nabla} F = \vec{E}$), also known as the "voltage." Then we conclude that the integrated electric field $\vec{E}$ in some region is the same as the surface integral of the voltage F around that region's boundary (times the infinitesimal outward-vector element of area).

There may be uses of these theorems when constructing "finite element" computer software for numerical solution of Maxwell's equations. I.e., by using these theorems, it would be possible to guarantee that the computer's approximate solution obeyed certain conservation laws *exactly*.

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