Somewhere on the sun, light does not shine out

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Abstract — **Work during 1870-1970 had shown that spherically symmetric electromagnetic, gravitational, or massless-neutrino energy flows were impossible. These impossibilities were thought by their discoverers to be isolated results dependent on the field equations obeyed by these kinds of energy. But we now show that they are consequences of extremely general and simple topological impossibilities, and have little or nothing to do with the field equations. Consequently we get more general impossibility results from shorter and simpler proofs. We also get a complete classification of the allowed topologies of electromagnetic, gravitational, or neutrino wavefronts – only planes, infinite cylinders, and tori are allowed.**

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This paper will say something about some highly advanced theories of physics, but it will do so in a wholy elementary manner, so fear not.

Electromagnetic fields. About 3.8×10^{26} watts of electromagnetic power shines out through any reasonably small sphere containing the sun. Nevertheless

Theorem 1: *Let S be a surface diffeomorphic to a sphere. Let $\vec{E}$ and $\vec{B}$ be (possibly space and time dependent) continuous electric and magnetic fields. Then, somewhere on S, the electromagnetic energy flux is not directed outward.*

Proof. The electromagnetic energy flux [7] is proportional to the Poynting vector $\vec{P} = \vec{E} \times \vec{B}$. If $\vec{P}$ had a nonzero outward component everywhere on S, then $\vec{E}$ would necessarily have a nonzero component tangential to the surface of S, everywhere. But it is impossible to "comb the hair on a sphere," i.e. there cannot be a nonvanishing continuous vector field on a sphere [3][9]. □

Remarks. Of course, this theorem still works if S is allowed to distort at sublight speeds, or if the word "outward" is replaced by "inward," or if spacetime is an arbitrary smooth $(3 + 1)$-manifold (as in General Relativity) rather than just flat space. Also, we may conclude that somewhere on S the energy flux must be directed neither inward nor outward. A simple oscillating-dipole radiator suffices to cause outward energy flux everywhere except at the North and South poles of a sphere (e.g. see EQs 9.19 and 9.23 of [7]).

In fact it is possible to set up an electromagnetic field on a sphere which is radiating outward everywhere except at a *single* point: in an xy plane make two vector fields which point parallel to the positive x and positive y axes, and then conformally map these two fields to the surface of a sphere via the stereographic projection. Indeed, the domain of this field may be extended to a solution of Maxwell's vacuum equations on the whole of spacetime (except for one spatial point), as follows. Start with a solution of the Maxwell equations representing a plane wave propagating in the z direction. Now perform a Cunningham transformation [2] of spacetime. This transformation converts the xy plane into a sphere S, and preserves solutions of the Maxwell vacuum equations. Thus we generate an outward (or inward) propagating spherical wave – but which is neither outward nor inward propagating at a single point on S. I do not know if this mathematically constructed solution corresponds to something physically important. (For some other recent theorems in elementary electromagnetism and vector calculus, see [13].)

To the physicist who objects that the light shining out of a spherically symmetric star certainly *seems* spherically symmetric and time-invariant, we reply that the light is "incoherent" radiation, which, examined on micron length scales and femtosecond time scales, is a very wiggly and random-looking electromagnetic field which is not at all spherically symmetric nor steady! Our theorem says, essentially, that "cleaning up" this radiation field to make it beautifully symmetric ("building a spherically symmetric laser") is not possible.

Dirac electron fields. What about other (non-electromagnetic) forms of radiation? It is easy to produce spherically symmetric examples showing that a Dirac electron field *can* shine outward everywhere. In fact, Feynman's "propagator," or "Green's function," representing "what an initially stationary electron at the origin would do" [12], is an example, if integrated over an initial spherically symmetric 4-complex-component wave function of electrons

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such as the "helium ground state" wavefunction (the exact solution for a two opposite-spin electrons in a Coulomb potential; we are considering what would happen to the electrons if we suddenly "removed the nucleus").

Scalar fields trivially can shine outward everywhere, if the direction of shining is taken to be the gradient of the field.

Neutrino fields. Here we speak of the old-fashioned "2-component neutrino field" relevant for massless chiral lightspeed neutrinos. (Recently it was discovered that neutrinos actually have a nonzero, although very small, mass. Hence they presumably really travel at sublight speeds, can change chirality, and are described by the same sort of Dirac equation and 4-component wave function that describes electrons – or perhaps some more complicated equations. But we shall ignore that. Weyl's 2-component field was good enough to win Lee and Yang the Nobel prize, so it shall be good enough for us.) Despite the fact that the neutrino luminosity of the sun is comparable to its electromagnetic luminosity, it is known [4] that spherically symmetric pure-radiation neutrino fields are impossible. More strongly:

Theorem 2: *Let S be a surface diffeomorphic to a sphere. Let N be a field of left-helical neutrinos. Then, somewhere on S, the neutrino flux is not directed outward.*

Intuitive Non-Proof. The most striking characteristic of 2-component neutrinos is their "helicity." That is, they spin, and the direction of that spin is always oriented the same way with respect to their direction of motion – like a left-handed screw. Each neutrino carries a fixed amount ($\hbar/2$ in fact) of angular momentum.

Imagine that S is coated with a film of magic fluid[1] with the following property. Whenever a neutrino hits it, it instantly absorbs the neutrino and emits something spinless with the same momentum as the neutrino. (This is exactly what would happen in a head-on lossless collision of two frictionless equal mass billiard balls.) In that case the fluid will absorb only the *angular* momentum of incident neutrinos. The fluid will then immediately acquire a velocity field tangent to S having leftward rotation everywhere. But there is no velocity field on a sphere with leftward rotation everywhere! (One might call this the "hurricane lemma" – it is impossible to have a planet-wide everywhere-leftward-rotating hurricane – and regard it as a remarkable fact of topology analagous to the hairy ball theorem. But in fact, this is just an immediate consequence of Stokes' theorem from vector calculus: if such a velocity field existed, an obviously-positive surface integral would have to be zero – contradiction.) □

This non-proof sounds convincing, and if correct would apply to *any* kind of field with helicity. But, sadly, it is incorrect, because it would (more strongly than the theorem says!) "prove" the impossibility of having *positive* neutrino flux out of the sun. A genuine proof based on 2-spinors will be given later.

Gravitational radiation. Gravity is complicated by the fact that there is no known tensor giving the local energy density and flux of a gravitational field. But in the *linearized* theory of gravity, approximately valid when space is nearly flat, there is an energy *pseudo*tensor [10]. It is estimated (p.266 of [16]) that thermal collisions in the sun radiate about 10^8 watts of gravitational wave energy. Nevertheless

Theorem 3: *Let S be a surface diffeomorphic to a sphere. Let $h_{\alpha\beta}$ be a linearized gravitational wave. Then the wave energy flux cannot be outward-flowing everywhere on S.*

Proof. We are considering a small perturbation (the "wave") $h_{\alpha\beta} = h_{\beta\alpha}$ of the metric tensor of flat spacetime

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

As is explained on pages 946-955 of [10], we may without loss of generality assume that $h_{\alpha\beta}$ satisfies the 8 "transverse-traceless gauge" conditions that $h_{xx} + h_{yy} + h_{zz} \equiv 0$ and $h_{tt} = h_{tx} = h_{ty} = h_{tz} \equiv 0$ (and 3 other conditions we shall not need here), i.e. $h_{\alpha\beta}$ is a symmetric traceless 3×3 matrix. As is also explained in [10] (p.951), the physical effect of such a wave is to produce oscillations in the separation of nearby point-pairs, and only separations in directions *transverse* to the direction of wave propagation can oscillate. (The "direction of wave propagation" is determined at a point from the first derivatives of the $h_{\alpha\beta}$ there.) Indeed ([10] p.952), when restricted to a 2D plane perpendicular to the direction of motion of the wave, $h_{\alpha\beta}$ is a 2×2 symmetric traceless matrix (call it M). So M has two mutually perpendicular eigendirections within that 2D plane. (For facts from linear algebra, see [6].) Suppose M is *nonzero*. Then since M is traceless, its eigenvalues are distinct reals, and hence M's two eigendirections are unique. If the wave were, everywhere on S, traveling exactly perpendicular to S, these eigendirections would everywhere be tangent to S. So if M is nonzero everywhere on S (except perhaps on a subset of zero spatio-temporal measure), then either one of these eigendirections would form a continuous direction field on S. But by the hairy ball theorem, that is impossible. (Variants [15] of the hairy ball theorem actually rule out any continuous nonzero *bi*direction field on the sphere – i.e., a unit-vector field where vectors are identified with their negation – i.e. a "foliation by

[1] To any physicist who complains that our "magic fluid" is unavailable in local drugstores, we point out that all that actually is required is a collection of lots of little neutrino detectors accompanied by light-emitting diodes, and it does not matter at all, logically, how inefficient those detectors may be, nor whether anybody ever actually builds one, so long as it is theoretically possible to do so.

1-manifolds.") On the other hand, suppose M is *zero* on an S-subset of nonzero spatio-temporal measure, then the wave would have zero amplitude there and would not be transmitting energy, in which case our theorem would be trivially satisfied. So the proof is now complete if we only speak of waves moving exactly perpendicular to S everywhere. Now permit the direction of wave propagation *not* to be exactly perpendicular to S, provided it still is everywhere outward-pointing. Then we can project the two eigendirections in the 2D wavefront planes down onto S locally everywhere, still getting a continuous direction field on S. Again, the hairy ball theorem does not permit this. □

Summary. All of our theorems have had the same form: the flux of energy of type X cannot be everywhere outward from any topologically-spherical surface. This holds when X is a massless spin-Z field for $Z = 1$ (electromagnetism) and $Z = 2$ (linearized gravity), and as we have not yet proven (but soon will) when $Z = 1/2$ (2-component neutrinos). In all cases the proof depends on the "hairy ball theorem." But it does *not* hold for $Z = 0$ (scalar fields), nor in the massive case when $Z = 1/2$ (electrons). It also does not hold in the massive spin-1 case (vector bosons) as the following makes clear: the potential between two spherically symmetric "balls of charge" r apart in a theory where "photons have mass" is a Yukawa potential $Qe^{-r/\ell}/r$ which in the Maxwell (massless photon) case degenerates to a Coulomb potential Q/r. If one of the charge-balls contracts, then the Coulomb potential remains unchanged outside of the original ball, *but* the Yukawa potential changes to a different Yukawa potential. The latter corresponds to a spherically symmetric transmission of energy. (The same argument also shows spherically symmetric energy transmission is possible for massive spin-2 fields.)

This leads us to **conjecture** that a version of our theorem holds for spin-$n/2$ *massless* fields for *any* positive integer n, with masslessness being essential. The rest of this paragraph describes what partial progress I've made toward rigorous formulation and proof of this conjecture. The beautiful theory of 2-spinors [11][14] enables expressing *all* the wave equations of spin-$n/2$ fields ($n \geq 1$) in a spacetime manifold in the following, wonderfully simple appearing, unified form:

$$\partial_{AA'}\phi^{\overbrace{}^{}\!\!\underbrace{ABCD...Z}_{2n \text{ indices}}} = 0. \tag{2}$$

There is a tremendous amount of complexity hidden inside this notation [11], which we shall not attempt to explain in the slightest. We instead shall merely point out the fundamental insight behind 2-spinors, which is that by regarding a 4-vector (u_0, u_1, u_2, u_3) as a 2×2 Hermitian matrix

$$M = \begin{pmatrix} u_0 + u_3 & u_1 - iu_2 \\ u_1 + iu_2 & u_0 - u_3 \end{pmatrix} \tag{3}$$

the scalar part of the matrix gives the time-component (i.e. $2u_0 = \text{trace}M$) while $\det M = u_0^2 - u_1^2 - u_2^2 - u_3^2$ gives the Lorentzian norm. EQ 2 is invariant under "conformal transformations" of the spacetime manifold [11]. Furthermore [11], for $n = 1, 2$ there is an energy-momentum tensor $T_{\alpha\beta}$ determined by the spinor field ϕ, which is, when $n = 1$, bilinear in ϕ and $\nabla\phi$. Unfortunately [11], for $n = 3, 4, \ldots$ there is *no* symmetric divergence-free energy-momentum tensor $T_{\alpha\beta}$ that is bilinear in ϕ and $\nabla\phi$ or quadratic in ϕ. (That rather detracts from our dream of formulating and proving a unified super-theorem. But we press on.) Because (as is well known) all diffeomorphic 2-manifolds are in fact *conformally* equivalent, we may by a conformal transformation reduce our problem to the case where our sphere is, in fact, the standard round 2-sphere. Furthermore, because no second or higher derivatives are involved (at least when $n \leq 2$) so that spacetime curvature is irrelevant, we may without loss of generality regard that standard round sphere as being embedded in ordinary *flat* $(3+1)$D spacetime. Because EQ 2 is *linear*, its complete solution may be written as a series in terms of "spin weighted spherical harmonics." It feels as though we have almost all the needed ingredients, and that there is a wonderfully simple unified theorem and proof lurking – but perhaps because of my relative unfamiliarity with 2-spinors, I'm unsure what to do next.

Let us now return to the $n = 1$ case of massless neutrinos, where we *can* make progress. The argument above has reduced the problem to "can neutrino energy shine out of a round sphere in flat space, everywhere?" The neutrino field is a single-indexed 2-spinor ϕ^A, which may be thought of as having 2 complex, or 4 real, degrees of freedom. From the 2-spinor viewpoint, it may be thought of as a field of 2×2 Hermitian matrices representing linear transformations in the 2D tangent spaces (extended from 2 real to 2 complex dimensions) of our sphere.

Lemma 4: *There does not exist a continuous 2-spinor field on a 2-sphere S, unless those 2-spinors are scalar (i.e. multiples of the 2×2 identity matrix) throughout some nonempty open subset of S.*

Proof. Consider each of the two (mutually orthonormal) eigenvectors of the matrices. These eigenvectors are uniquely determined wherever our matrix is nonscalar. Of course, eigenvectors are continuous functions of their matrices. Either eigenvector may be multiplied by a complex phase factor in an effort to maximize the L_2 norm of its real part, and this effort will always succeed in raising that norm above 0. Considering this real part then yields a "combing of hairs" on S everywhere except in the strict interior of the subset of S on which our matrices are scalar. □

By painful but straightforward brute force expansion into components using the known expressions for $T_{\alpha\beta}$ [1][11]) one may now confirm that a field of scalar 2×2 matrices on our sphere corresponds to a neutrino field that is not transmitting any energy radially. That proves theorem 2. □

Do our very simple arguments have significant consequences? We argue "yes." In order to satisfy our theorems and avoid outward shining somewhere, any spherically *symmetric* wave would have to avoid outward shining *anywhere*. Hence it would have to shine (i.e. transmit energy) only in directions tangent to concentric spheres, but spherical symmetry and the hairy ball theorem make that impossible. So our theorems immediately tell us that spherically symmetric nonzero fluxes of energy (of the sorts covered by our theorems) are *impossible*.

(A) It is known that spherically symmetric gravitational waves are impossible in the general relativity vacuum; that is an immediate consequence of "Birkhoff's uniqueness theorem" stating that the Schwarzschild metric (which is static) is the *unique* spherically symmetric general relativistic vacuum. Our theorem 3 is a generalization of Birkhoff's wave-impossibility theorem to *any* topologically spherical surface, and, since the equations of general relativity were almost irrelevant to the proof, it presumably now also applies to many other theories of gravity intended to go beyond general relativity.

(B) Hoffmann [5], in a partially successful effort to generalize Birkhoff's uniqueness theorem in general relativity to make it apply to solutions of the Einstein-Maxwell equations of gravity *and* electromagnetism in vacuum, proved that any spherically symmetric solution of the Einstein-Maxwell equations must have a *static* electromagnetic field. (Much later Lovelock [8] showed by example that there are at least *two* kinds of spherically symmetric vacuum solutions of the Einstein-Maxwell equations – both, of course, static – killing all hope of proving a full uniqueness theorem.) Hoffmann's staticity theorem follows immediately from our theorems 1 and 3. (Proof: any spherically symmetric combined gravitational and electromagnetic field would have to have all energy flux *tangent* to the surfaces of the concentric spheres, but by the hairy ball theorem no suitable nonzero flux field can exist. Hence the energy flux must be identically zero, so we have staticity. □) So here our dark sun theorem has enabled replacing a very complicated argument in general relativity with a short proof easily comprehensible by undergraduates.

(C) Griffiths and Newing [4], via a very complicated argument, proved the impossibility of a spherically symmetric 2-component neutrino pure-radiation field in general relativity. From our theorem 2 we immediately get a more general statement (e.g. one may substitute the words "non-static" for "pure-radiation") again with a trivial proof.

(D) In our impossibility theorems, instead of topological "spheres," we could use "k-holed tori with $k \geq 2$" since it is also impossible to comb hair on them, or to foliate them into 1-manifolds. (However, it *is* possible to have toroidally and cylindrically symmetric outgoing scalar, electromagnetic, massless neutrino, and linearized gravitational waves. The proofs are left as exercises for the reader – with the aid of a trick there is a common trivial solution.)

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