Pythagorean triples, rational angles, and space-filling simplices

Warren D. Smith `WDSmith@fastmail.fm`

20 Dec. 2003

Abstract

The ancient Greeks posed and solved the problem of finding all right triangles with rational sidelengths. There are 4 natural *non*Euclidean generalizations of this problem. We solve them all. The result is that the *only* rational-sided nonEuclidean triangle with one right angle is the isoceles spherical triangle with legs of length 45° and hypotenuse 60°.

We next ask which simplices have rational dihedral angles (measured in degrees). The solution is easy once connection is made to 1934 work of Coxeter. There are only a finite number of examples in 3-dimensional Euclidean space and only a countable number in n-space for $n \geq 4$, which are nowhere dense in the space of simplices. But there are a dense and infinite set of examples if $n = 2$ or in nonEuclidean n-spaces for each $n \geq 2$.

In contrast, there are a continuum infinity of n-simplex shapes which tile n-space and are equidecomposable with n-parallelipipeds, as we show by explicit construction of infinite familes of of simplex tilers of n-space, for each $n \geq 4$. (Some of these were previously known, while others are new.) There is a dense continuum infinity of n-simplex shapes with Dehn invariant 0.

Along the way we prove that "Plouffe's constant" and related angles are transcendental (Plouffe had not even known if they were rational).

Although these four problems seem unrelated, they in fact are related.

Contents

1 Pythagorean triples

A Pythagorean triple is three positive integers a, b, and c such that a right triangle exists with legs a, b and hypotenuse c, i.e. such that

$$a^2 + b^2 = c^2.$$

The simplest two Pythagorean triples are $(3, 4, 5)$ and $(5, 12, 13)$.

The problem of finding all Pythagorean triples was solved by the Ancient Greeks. They are generated by $(w^2 - u^2, 2uw, w^2 + u^2)$ where u and w are relatively prime ([31] p.141).

The Pythagorean triple problem has spawned a great deal of mathematics [13][33], the most famous example being the proof by A.Wiles [36] of "Fermat's last theorem" that

$$a^n + b^n = c^n$$

has no positive integer solutions if $n > 2$.

Both of these problems may be rephrased to concern *rational* a, b, c, since any rational solution generates an integer solution upon multiplying by the least common denomonator, while any integer solution generates an infinity of rational solutions by scaling it by any rational factor.

2 NonEuclidean Pythagorean triples

Our purpose is to generalize this problem in a different direction: toward nonEuclidean [11][14] geometry.

The nonEuclidean Pythagorean triple problem: *Find all triples (a, b, c) of positive rational numbers such that a right triangle exists, drawn on (I) the surface of the unit sphere or (II) the unit-negative-curvature hyperbolic plane (which may be regarded as "the sphere with radius $\sqrt{-1}$"), with legs a, b and hypotenuse c measured in either (1) degrees or (2) radians.*

This is actually 4 different problems I1, I2, II1 and II2. (For those who worry that degrees are "unnatural" since the number 360 was arbitrary, we remark that any other number x besides 360 would yield the same problem, since multiplication by $x/360$ preserves rational numbers.)

The nonEuclidean Pythagorean theorem tells us that the solutions precisely correspond to the triples (a, b, c) of positive rational numbers such that

$$\begin{aligned} \text{I1}: &\quad \cos(\pi a)\cos(\pi b) = \cos(\pi c) \\ \text{I2}: &\quad \cos(a)\cos(b) = \cos(c) \\ \text{II1}: &\quad \cosh(\pi a)\cosh(\pi b) = \cosh(\pi c) \\ \text{II2}: &\quad \cosh(a)\cosh(b) = \cosh(c) \end{aligned}$$

Without loss of generality we may demand $a \geq b$. We now solve the latter three problems completely.

Theorem 1: *There are no solutions of I2, II1, or II2 in nonzero rational numbers* a, b, c.

Proof: C.Hermite proved in 1873 that e is transcendental. A.Gelfond (and independently T.Schneider) proved in 1934 that e^{π} is transcendental. Finally, by the Hermite-Lindemann theorem, $\sin(1)$ and equivalently e^{i} are transcendental ([19] p.160, [2][25][32]). If any nontrivial rational solutions (a, b, c) of the above equations existed, then (by expanding the cosh or cos into exponentials) that would yield an algebraic relationship satisfied by one of these three transcendentals, a contradiction. Q.E.D.

We now consider the spherical problem I1. We may without loss of generality demand that all the arguments of the cos()'s lie in $(0, \pi/2]$. That is because by continuing the sides of a spherical triangle to become full great-circle geodesics, we get the 8 versions of any spherical triangle. Only the one of these with the smallest sidelengths is of interest, since the other 7 may be regenerated from it.

The *doubly right angled isoceles* triangles with apex at the North pole and base (of any rational length b) on the equator yield a trivial set of solutions of I1 with $a = c = 1/2$. These may be excluded by only permitting triangles with a *single* right angle.

The solution of problem I1 is:

Theorem 2: *Aside from the trivial solutions* $(1/2, b, 1/2)$*, there is exactly one solution of I1 in rationals* $a, b, c \in (0, 1/2]$*, namely* $(1/4, 1/4, 1/3)$.

The proof is deferred until §5. Nothing like the preceding proof technique will suffice because all the quantities in I1 are algebraic numbers.

The unique solution corresponds to the isoceles spherical triangle with legs of length 45 and hypotenuse 60 (measured in degrees), apex angle 90°, and both[1] base angles $\arctan(\sqrt{2}) = \mathrm{arcsec}(\sqrt{3}) \approx 54.73^{\circ}$. This triangle is, in fact, one quarter of a square face of the cuboctahedron (Archimedean semiregular polyhedron [10] with 8 square and 6 triangular faces, two of each alternating in cyclic order around each vertex, and with all sidelengths equal).

Before proving anything, we remark that a computer verified theorem 2 among all (a, b, c) such that at least two of their denominators are ≤ 60 and such that the remaining denominator is $\leq 10^{25}$. The technique was to exhaustively consider the pairs of rationals with denominator≤ 60; for each the remaining number was computed to a large number of decimal places and it was verified (by computing its regular continued fraction) that it was not a rational number with denominator$\leq 10^{25}$. A different computer run verified that $\arccos[\cos(\pi/m)\cos(\pi/p)]/\pi$ is apparently irrational (has denominator $> 10^{50}$) for all integers $3 \leq m \leq p \leq 360$.

3 The transcendence of certain angles

We now turn to a seemingly unrelated problem. Define the "unit step function" $u(x)$ to be 0 if $x \leq 0$ and 1 if $x > 0$. Now define a_n to be the nth iterate of $x \to 2x/(1 - x^2)$ starting from $a_0 = 1/2$. Then Simon

[1] This base angle is half the tetrahedral "carbon bond angle." By theorem 4 using the fact that $2/\sqrt{3}$ is not an algebraic integer – or one could instead proceed from theorem 5 using the fact that $\arctan\sqrt{2} = \langle 3\rangle_2$ – it is transcendental measured in either degrees or radians.

solid	#vertices	n-volume	angle (& CRS form)
tetrahedron	4	$\sqrt{2}/12$	$\operatorname{arcsec}(-3) = 2\langle 3\rangle_2 \approx 109.47°$
octahedron	6	$\sqrt{2}/3$	$\pi/2 = 90°$
cube	8	1	$\operatorname{arcsec}(3) = \pi - 2\langle 3\rangle_2 \approx 70.53°$
icosahedron	12	$5\tau^2/6$	$\pi - \arctan 2 = \pi - \langle 5\rangle_1 \approx 63.435°$
dodecahedron	20	$\sqrt{5}\tau^4/2$	$\arcsin(2/3) = \pi/2 - 2\langle 3\rangle_5 \approx 41.81°$
regular 4-simplex	5	$\sqrt{5}/96$	$\operatorname{arcsec}(-4) = \pi - 2\langle 2\rangle_{15} \approx 104.48°$
4-octahedron	8	$1/6$	$\pi/2 = 90°$
4-cube	16	1	$\pi/3 = 60°$
600-cell	120	$25\tau^3/4$	$\pi/5 = 36°$
120-cell	600	$15\sqrt{5}\tau^8/2$	$\operatorname{arcsec}4 - \pi/3 = 2\langle 2\rangle_{15} - \pi/3 \approx 15.522°$
regular n-simplex	$n+1$	$2^{-n/2}\sqrt{n+1}/n!$	$\operatorname{arcsec}(-n)$
n-octahedron	$2n$	$2^{n/2}/n!$	$\pi/2 = 90°$
n-cube	2^n	1	$\pi - \operatorname{arccsc}\sqrt{n}$

Table 1: The regular polytopes in 3, 4, and $n \geq 5$ dimensions [10]. For each we first give the n-volume (for polytopes with unit edge length), then the smallest intervertex angle as viewed from its center, both as a formula, approximately in degrees, and expressed in CRS canonical form [6]. We shall prove all these angles are transcendental when measured either in radians or in degrees (except for the ones, all shown, which are an integral number of degrees; these are still transcendental in radians). When $n = 3, 4$ we shall also prove that any *ratio* of two of these angles is either transcendental or rational (the latter only when both angles are an integral number of degrees). $\tau \equiv (\sqrt{5} - 1)/2 \approx 1.618$.

Plouffe found the following remarkable formula for the binary expansion of

$$C \equiv \frac{1}{\pi} \arctan 2 = \sum_{n \geq 1} \frac{u(a_n)}{2^{n+1}}.$$

Plouffe confirmed this formula numerically to a large number of decimal places, but its proof had to wait for Borwein and Girgensohn [4] in 1995. All these authors conjectured, but could not prove, that this number was irrational.

Calling C "Plouffe's constant" is somewhat misleading, since this angle had arisen in the geometry of the icosahedron (dating back at least to the Ancient Greeks) see table 1. Regarding such binary expansions as due to either Plouffe or Borwein & Girgensohn is also misleading, since the essential idea behind them was already known as the "CORDIC" algorithms for computing arctrig functions [34] (invented at least as far back as 1959 and arguably known to Archimedes) and since then the subject of numerous papers [35] and implemented inside numerous commercial electronic calculators (e.g. the HP-35), etc.

We shall now give a new proof[2] that Plouffe's constant, and indeed all the angles in table 1 and a wide class of other angles, are not only

[2]Simon Plouffe pointed out to me that Barbara Margolius [26] had independently proven (somewhat before I did) that Plouffe's constant was transcendental. She indeed showed (as

irrational, but in fact transcendental, when measured either in radians or in degrees (except for the angles such as 90° which are an integer number of degrees!).

Lemma 3. *If r is rational, then $X = 2\sin(\pi r)$ [and hence $2\cos(\pi r)$] is an algebraic integer, that is, the root of a monic polynomial with integer coefficients. In other words, doubling the sine (or cosine) of a rational angle (measured in degrees) yields an algebraic integer.*

Proof. By the cosine multiple angle formula, X is rational if and only if $U_n(X/2) = 0$ where n is an integer and U_n is the nth degree Chebyhev polynomial of the second kind [1]. $U_n(x/2)$ is a monic polynomial with integer coefficients because it is the determinant of a tridiagonal $n \times n$ matrix with x's in all diagonal entries and 1's in all super- and sub-diagonal entries. Q.E.D.

Now it is a well known lemma that the quadratic algebraic integers of the form $a + bz$, where a, b are integers and z is an irrational quadratic algebraic, are precisely those with $z = \sqrt{D}$ with D an integer with $D \equiv 2$ or 3 (mod 4) and or $z = (1 - \sqrt{D})/2$ with D an integer with $D \equiv 1$ (mod 4). (This conclusion also is what one would have thought directly from the quadratic formula for solving $x^2 + px + q = 0$.) Even more easily, a rational number is an algebraic integer if and only if it is an ordinary integer.

Thus, for example, the golden ratio $(\sqrt{5} - 1)/2 \approx 0.61803$ is an algebraic integer. But $2/\sqrt{5}$, $4/3$, $2/n$ for $n > 2$, $2/\sqrt{n}$ for $n \neq 1, 4$, and $1/2$ are not, from which it follows that all the angles in table 1 (and hence also Plouffe's constant) are necessarily irrational when measured in degrees, except for the ones that are an integer number of degrees.

Now, let X be any inverse trig function, inverse hyperbolic function, or natural logarithm, applied to an algebraic number – or any ratio of two such. The **Gelfond-Schneider theorem** [2][28][32] states that X is either rational or transcendental[3]. **Baker's extension** states that any "linear form in logarithms"

$$\sum_{k=1}^{n} \alpha_k \ln(\beta_k)$$

where the α_k and β_k are algebraic numbers such that each sumand is nonzero, is either rational or transcendental. Of course again, here, any ln's may be replaced by inverse trig or inverse hyperbolic functions. Three immediate corrollaries (in increasing order of complexity) are

1. that π is transcendental,
2. that any nonzero angle with an algebraic-valued trig function is transcendental when measured in radians, and

does [28], corollary 3.12 p.41) that $\arctan(r)$ is transcendental – measured in either degrees or radians – if r is rational and $r \notin \{-1, 0, +1\}$. Her interesting irrationality proof (in degrees) totally avoids using algebraic number theory and instead is based on a direct construction of good rational approximations – using properties [13] of Pythagorean triples!

[3] Warning: One cannot just use Baker's theorem to prove transcendence unless one removes the possibility of rationality first. This is in fact the major difficulty. Miraculous algebraic relationships such as $16\arctan(1/5) - 4\arctan(1/239) = \pi$ can destroy many a "proof" of transcendence.

3. is either rational or transcendental when measured in degrees.

But if $2\sin(\pi r)$ is not an algebraic *integer*, then r cannot be rational in case 3, and hence must be transcendental. We conclude:
Theorem 4: *If $2X$ is a nonzero algebraic number, then* $\arcsin X$ *is transcendental if measured in radians; if, further, X is not an algebraic integer, than* $\arcsin X$ *is also transcendental if measured in degrees.*

Conway, Radin, and Sadun [6] considered the set of "CRS-angles," i.e. those expressible as rational linear combinations of angles with rational squared trigonometric function values. The **Conway-Radin-Sadun theorem** states that

1. The CRS-angles are the same as the angles θ with $\tan\theta$ "polyquadratic," i.e. expressible as a finite sum of square roots of rational numbers $\sum_k \sqrt{r_k}$.
2. There is a certain countably-infinite nonredundant basis over the rationals for the CRS-angles, given in [6].

That is, every CRS angle is expressible *uniquely* as a rational linear combination of angles from the CRS basis.

The CRS basis angles are π and certain quantities (defined in [6]) denoted $\langle p \rangle_d$ for prime $p \geq 2$ and squarefree integers $d > 0$ obeying certain congruence conditions. From this it follows that
Theorem 5: *The ratio α/β of two CRS angles is always transcendental unless both have the exact same unique C-R-S representation (up to overall multiplication by some rational), in which case it is rational.*
Proof. The uniqueness theorem for CRS representations proves irrationality, and once we know α/β is irrational we then automatically know it is transcendental by the Gelfond-Schneider-Baker theorem. Q.E.D.

The claims in the caption of table 1 are now proven.

4 Simplices with rational dihedral angles (in degrees)

We now turn to another seemingly unrelated geometric diophantine problem. It was solved by H.S.M.Coxeter [8] in 1934, although he and his expositors [16][22] did not phrase it as a Diophantine problem at all, and hence the following discussion, which reinterprets this Diophantinely, is new. An "n-simplex" is the convex hull of $n+1$ points in n-dimensional space.
The Rational Simplex Problem: *What are the $\binom{n+1}{2}$-tuples of rational numbers in $(0,1)$ such that a n-simplex can exist with those, times π, as its dihedral angles?* Call such a simplex a "rational simplex."

This problem is trivial if $n = 2$ (any positive rational numbers a, b, c with $a + b + c = 1$ will do) or in nonEuclidean spaces (there are only *in*equality restrictions on the dihedral angles of a nonEuclidean simplex) or if we restrict attention to, e.g., hyperbolic tetrahedra with all vertices at ∞ (the 3 dihedral angles at each vertex must sum to π, while the sum of any 4 angles forming a 4-cycle of edges must exceed π [21]). However, the problem is nontrivial in Euclidean n-space with $n \geq 3$. As a non-example, the dihedral angle of the regular n-simplex, $n \geq 3$, is (by our preceding two theorems) transcendental measured in either radians or degrees.

Although it is not usually described this way, the complete solution of this problem was found by Coxeter while classifying "reflection groups." Observe that a rational n-simplex can be used to tile all of n-space by repeatedly reflecting it in its face-hyperplanes. More precisely this "tiling" will, in general, be a multiple covering, sometimes an infinitely-thick one. The essential thing about this tiling, or multiple covering, is that it is invariant under any of those face-plane reflections and thus defines a finite group of the reflections that preserve one of the simplex's vertices (which, without loss of generality, we may make the origin). If any dihedral had been an *ir*rational multiple of π, though, then the reflection group would be *infinite*. Coxeter *classified all finite reflection groups* defined by a finite set of origin-containing mirror-hyperplanes [8][16][22].

Each reflection group is defined by its set of mirror hyperplanes and the dihedral angles among them. The basic groups are known by the names A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, H_3, H_4, and $I_2(m)$. The subscript in each name gives the dimensionality. Each of these reflection groups is described by one of these diagrams:

```
        1   2   3         n             1   2   3         n
     A  o---o---o-- ... --o          B  o---o---o-- ... --o
      n                               n    4

       1 o                                          H
         |  4         n                              4
   D   3 o---o-- ... --o     H  o---o---o      5
    n    /                    3    5          o---o---o---o
     2 o

   I (m)  o---o    F  o---o---o---o     E  o---o---o---o---o
    2       m       4       4            6         |
                                                   o

               o                           o
               |                           |
   E   o---o---o---o---o---o      E   o---o---o---o---o---o---o
    7                              8
```

Figure 1. Coxeter groups.

Each node in each diagram represents an origin-containing face of the simplex. Two nodes not joined by an arc represent a 90° dihedral angle; if they are joined by an arc labeled by an integer m that represents a $(180/m)^\circ$ dihedral angle, and the implied label on an unlabeled arc is $m = 3$, meaning a 60° dihedral. It is also possible to combine these diagrams to get a graph with more than one connected component, where each such component is one of our diagrams (repeating a diagram is allowed). **Coxeter's classification theorem:** *The resulting set of n-node diagrams corresponds to the complete set of finite reflection groups in n dimensions* ([22] theorems in §6.4 p.133 and 2.7 p.36).

Although any simplex with rational dihedral angles (in degrees) yields a finite reflection group, the reverse is not necessarily the case: a finite reflection group will only necessarily yield a simplex all of whose dihedral angles *containing the vertex at the origin* are rational. Thus we actually get an over-enumeration[4] of rational simplices.

The point I want to make, which I have not seen mentioned before in spite of its importance, is the following (which we have just proven):

Theorem 6. *Whereas in the 2D Euclidean plane, triangles with rational angles (measured in degrees) are* dense *in the set of all triangles*[5]*, in Euclidean n-space with $n \geq 3$, the n-simplices with rational dihedral angles are nowhere dense in the set of all n-simplices. Indeed in 3D, only a finite number of rational tetrahedra exist, described by Coxeter diagrams, and all of their dihedral angles are multiples of 3°.*

If, however, the simplices are allowed to lie in a *non*Euclidean geometry, then we have density in all dimensions.

5 Grand unification

Now, finally, we prove theorem 2, solving problem I1, by making connection to the Coxeter rational-simplex problem. Consider a tetrahedron with all 6 of its dihedral angles rational (measured in degrees) and with 4 of those 6, forming a 4-cycle of edges, having *equal* dihedral angles s. This situation may be represented by placing 4 points (corresponding to the outward normal unit vectors to each of the 4 tetrahedron faces) on the unit sphere, and regarding them as the corners of a quadrilateral drawn on that sphere. We demand that the 4 sides of this quadrilateral all be equal in length to some rational number s (measured in degrees). This spherical quadrilateral, since all of its sides are equal, might more properly be called a "rhombus." The two diagonals of this spherical rhombus then are also required to have rational lengths (measured in degrees) since they correspond to the other two dihedral angles of the tetrahedron. But now, notice that these diagonals dissect the rhombus into four identical spherical right triangles! So, the spherical right triangles with all sidelengths rational are in 1-to-1 correspondence with the tetrahedra of our sort.

Then, by Coxeter's classification theorem in *three* dimensions (and our computer enumeration mentioned at the end of §2, we conclude that exactly one such tetrahedron – the A_3 tetrahedron – and hence exactly one such right triangle exist (this is provided we agree to ignore the doubly-right triangles). The proof of theorem 2 is now complete.

It is interesting that it seems much more difficult to attack problem I1 with algebraic number theory. (Indeed, using number theory I was unable even to show that there are only a finite number of rational triples solving I1.) Success ultimately came from geometry and group theory. Number theory is not always the best tool for solving number theoretic problems!

[4]But with the caveat that a finite number (perhaps exceeding 1) of simplices could conceivably lead to the same group.

[5]Parameterized by their dihedral angles.

6 Equidecomposability, and (new) simplices tiling n-space

J.H.Conway observed to me that any rational tetrahedron automatically is "equidecomposable with a cube," i.e. may be cut into a finite number of pieces via plane cuts, such that those pieces could then be rearranged to form a cube.

But there exist non-rational tetrahedra equidecomposable with a cube, and there exist tetrahedra (in particular, the regular tetrahedron) *not* equidecomposable with a cube [3].

Since there are only a countably-infinite number of rational n-simplices (and when $n = 3$ only a finite number) we are naturally led to ask how many n-simplices there are which are equidecomposable with a n-parallelipiped. In 2D, 3D, and 4D, this is known to be the same thing as equidecomposability with a *cube*, and also the same thing as having Dehn invariant 0.

Theorem 7: *For each $n \geq 2$, the n-simplices with Dehn invariant* 0 *are everywhere dense in the set of n-simplices, and there are a continuum infinity of them. However, for each $n \geq 3$, only a zero-measure fraction of all n-simplices have Dehn invariant* 0*.*

Proof sketch: The Dehn-Sydler-Jessen theory of equidecomposability, and Hilbert's third problem, are discussed in [3]. Max Dehn showed in 1902 that a *necessary* condition for equidecomposability of two n-polytopes of equal volumes is that their "Dehn invariants" be equal. The Dehn invariant of an n-polytope is the sum, over all $(n-2)$-flats of that polytope (i.e., if $n = 3$, the *edges* of that polyhedron), of the dihedral angle at that flat (measured in degrees) *modulo* the rational numbers $\mathbf{Q}$, times the measure of that flat (i.e., if $n = 3$, the edge length). Thus the Dehn invariant of a regular tetrahedron with 6 edges of length ℓ is

$$(\frac{180}{\pi}\operatorname{arcsec}(3) \bmod \mathbf{Q}) \cdot 6\ell,$$

and the Dehn invariant of any n-parallelipiped is 0. Since $\operatorname{arcsec}(n)/\pi$ is transcendental, we see that the regular n-simplex, $n \geq 3$, is not equidecomposable with any n-parallelipiped "brick." That behavior therefore is generic, immediately showing the zero-measure claim of the theorem. (Also, it is easy to see that any space-tiling simplex must have Dehn invariant 0.)

When $n = 2$, *any* two equal-area polygons are equidecomposable (shown by Farkas Bolyai and P.Gerwien in the early 1800s; [3] indeed proves this even in a nonEuclidean plane). When $n = 3$, J-P.Sydler showed in 1965, and when $n = 4$ B.Jessen showed in the late 1970s [3], that Dehn's criterion was also *sufficient.*

It is obvious from the definition of Dehn invariant that any sufficiently-generic smooth 1-parameter family of n-simplices (for any fixed $n \geq 2$) must have Dehn invariant crossing 0 infinitely many times in any finite parameter interval. Thus, n-simplices with Dehn invariant 0 are dense in the space of all n-simplices (and by the classic works mentioned in the preceding paragraph, if $n = 2, 3, 4$ any such simplex necessarily is equidecomposable with an n-cube).

Now, since there are a continuum infinity of disjoint smooth generic 1-parameter families of simplices, it follows that for each $n \geq 2$ there are a continuum infinity[6] of n-simplices with Dehn invariant 0. Q.E.D.

In retrospect it is easy to understand why simplices of Dehn invariant 0 are so much more common than rational simplices. Roughly speaking, the former simplices need to obey only a single rationality constraint, whereas the latter need to obey about $\binom{n+1}{2}$ of them.

The preceding result, while satisfying, was proven in an entirely non-constructive manner. Many people will prefer a proof which actually constructs an explicit continuum-infinity of n-simplices!

When originally writing this paper I was unaware that Debrunner [12] had already constructed $\lfloor d/2 \rfloor + 2$ one-parameter families of d-simplices which tile d-space and have Dehn invariant 0. The case $d = 3$ of space-tiling tetrahedra is reviewed by Goldberg [15] and Senechal [30]. I re-discovered some of Debrunner's space-filling simplices, but in a different way.Let $\phi(d)$ be the Euler totient function [1]. Upon combining my methods with Debrunner's ideas, the final result is:

Theorem 8: *For each $d \geq 2$, there are at least $(\lfloor d/2 \rfloor + 2)\phi(d)/2$ one-parameter families of d-simplices which tile d-space. (In some cases mirror-image tiles are required, in others not.) Any of these simplices has the property that a finite number of congruent copies of them, in net, are equidecomposable with a d-parallelipiped. However, if $d \geq 3$ then only a measure-0 subset of all d-simplices have either of these properties.*

Proof. First of all, any triangle tiles the plane and is equidecomposable with a square [3]. So assume $d \geq 3$ from now on.

Goldberg [15] noted that a 1-parameter family of tetrahedra, all of which tiled 3-space, could constructed as follows. Start with the tiling of the ("horizontal") plane by equilateral triangles of unit edge length[7]. Take the cartesian product of this with the real line to get a tiling of 3-space by congruent vertically-infinite equilateral-triangular prisms. Now each prism may be divided into congruent tetrahedra with edge lengths $3a$,b,b,b,c,c, where $b^2 = a^2 + 1$, $c^2 = 4a^2 + 1$, and a is arbitrary. This is done as follows: the prism is cut by a plane which intersects the 3 vertical infinite edges at heights na, $(n + 1)a$, and $(n + 2)a$. The next cut-plane does the same – but with $n + 1$ in place of n and the 3 edges rotated so that the lowest two edge-cut points of the $(n + 1)$th plane coincide with the highest two edge-cut points of the nth plane. The meaning of the free parameter a may be understood as follows: Once such a division of a prism into an infinite stack of congruent tetrahedra is found, all oriented in the same way up to rotations coaxial with the prism, the prism then may be arbitrarily "stretched" along its axis while preserving the congruences among the tetrahedra. Every member of this family of tetrahedra necessarily has Dehn invariant 0 and is equidecomposable with a cube. (Goldberg also noted that two other infinite families of tetrahedral space-tilers could be produced from his by [i] splitting his tetrahedron by

[6] In fact, this shows there should be p-parameter families of n-simplices with Dehn invariant 0, where p is one less than the number of parameters regarded as describing the entire space of n-simplices.

[7] For later reference, we note that the equilateral triangle is also known as the Coxeter 2-simplex $A_2 = I_2(3)$.

bisecting its length-$3a$ edge [ii] splitting it by instead bisecting the *middle* length-b edge, where the three length-b edges form a 3-edge path which together with the length-$3a$ edge forms a quadrilateral.) The maximally symmetric member of this family has $3a = b = 3/\sqrt{8}$ and coincides with the Coxeter simplex A_3. That tetrahedron's 6 edges are as follows: Four edges, each having a 60° dihedral angle, form a 4-cycle; and the remaining two disjoint edges each have 90° dihedrals. All 4 triangular faces of the A_3 tetrahedron are congruent.

We shall now explain how to generalize Goldberg's construction to 4D, 5D, and so on. This constitutes a rediscovery of the 1-parameter family that Debrunner [12] calls the "Hadwiger-Hill family $H^d(w)$" of d-space-tiling simplices. As Debrunner [12], Senechal [30], and Boltyanksii [3] pointed out, in fact Goldberg's 3 families had been found much earlier by Hill in 1896 [20]. Hill realized they each were equidecomposable with a cube. Goldberg's *true* contribution was to realize, via his tiling of infinite triangular prisms, that these tetrahedra each tile 3-space *without* need for any mirror image tiles. But Goldberg's tiling in general is not face-to-face, whereas a tiling which employs mirror image tiles can be done in a face-to-face manner. A generalization of Hill's construction to d dimensions was done by Hadwiger [17]. Then Debrunner showed how to turn Hadwiger's family into $\lfloor d/2 \rfloor + 2$ families by means of various "halving" and "doubling" operations. Our rediscovery of the Hadwiger-Hill family in a *different way* (following Goldberg's ideas instead of Hadwiger and Hill's) has the advantages that (i) it makes it obvious that there are really $\phi(d)/2$ *different* families, and (ii) it makes it obvious that each of them tile d-space without need for mirror image tiles. However, this tiling is (in general) not face-to-face. A face-to-face tiling can be done if mirror-image tiles are used.

The new construction makes it obvious that each of Debrunner's halved and doubled variants of $H^d(w)$ come in $\phi(d)/2$ flavors. One reason this was not previously noticed is that $\phi(3)/2 = \phi(4)/2 = 1$ so that no additional simplices arise in the heavily studied 3D and 4D cases. However, $\phi(5)/2 = 2$ so that we double the number of known families of 5-simplex space-tilers. Further, in arbitrarily large dimensions d, the factor $\phi(d)/2$ becomes arbitrarily large. (E.g. $\phi(p) = p - 1$ if p is prime, $\phi(2^n) = 2^{n-1}$, etc.)

We now explain the 4-dimensional construction. Start with the tiling of $\mathbf{R}^3$ by Coxeter A_3 simplices. (The vertices of these simplices form the body-centered cubic point lattice A_3^*, consisting of the integer 3-vectors with all coordinates even or all odd.) Take the cartesian product with the real line to get a tiling of 4-space by congruent infinite prisms each of whose bases is an A_3 simplex. Now cut those infinite prisms into 4-simplices with hyperplanes as follows: Hyperplane n intersects the 4 infinite edges of the prism at heights na, $(n+1)a$, $(n+2)a$, $(n+3)a$, using those edges in the same cyclic order as is induced in the vertices of the base A_3 tetrahedron by its 4-cycle of edges with 60° dihedrals. The lowest 3 edge-cut vertices induced by hyperplane $n+1$ coincide with the highest 3 edge-cut vertices induced by hyperplane n. Thus we have produced a 1-parameter family of 4-simplices which tile $\mathbf{R}^4$.

Here is how to make a 5D version of this contruction. Start with the tiling of 4-space by A_4 simplices. The vertices of these simplices form the "A_4^* lattice" ([7] p.114). In each A_4 simplex: All faces (and all vertex-neighborhoods) are congruent. Each vertex is attached to $6 = \binom{4}{2}$ triangles, 3 of which have a 60° and 3 of which have a 90° dihedral; the 6 edges they induce on the far face (which is a tetrahedron) are a 3-edge (i.e. 4-vertex) path and a disjoint 3-edge path. Thus each of the 5 vertices of A_4 may regard itself as having 2 special neighbor vertices, namely, the two vertices of the far face which are the endpoints of the 3-edge path arising from 60° dihedrals. The graph of "special neighbor" relations forms a 5-cycle. (Alternatively, one could use the path of 90° dihedrals. Thus there are at least two equally valid constructions here.) All 5 tetrahedral faces of the A_4 simplex are congruent. Now, take the cartesian product with the real line to get a tiling of 4-space by congruent infinite prisms each of whose bases is an A_3 simplex. Now cut those infinite prisms into 4-simplices with hyperplanes as follows: Hyperplane n intersects the 5 infinite edges of the prism at heights na, $(n+1)a$, $(n+2)a$, $(n+3)a$, $(n+4)a$ using those edges in the same cyclic order as is induced in the vertices of the base A_4 simplex by its 5-cycle of "special neighbor" relations. The lowest 4 edge-cut vertices induced by hyperplane $n+1$ coincide with the highest 4 edge-cut vertices induced by hyperplane n. Thus we have produced a 1-parameter family of 5-simplices which tile $\mathbf{R}^5$.

It should now be clear how to continue this construction into arbitrary dimension $d \geq 3$. Start with the tiling of $\mathbf{R}^{d-1}$ by Coxeter simplices A_{d-1}. The vertices of those simplices form the point lattice A_{d-1}^* ([7] p.114). Take the Cartesian product of this (horizontal) tiling with the vertical real line to get a tiling of $\mathbf{R}^d$ by congruent infinite simplicial prisms, each of whose bases is an A_{d-1} simplex. Now, partition each of the infinite prisms into congruent d-simplices by cutting them with hyperplanes as follows. The nth hyperplane intersects the d infinite edges of the prism at heights na, $(n+1)a$,..., $(n+d)a$ where a is a free parameter. The lowest d edge-cut vertices induced by hyperplane $n+1$ coincide with the highest d edge-cut vertices induced by hyperplane n. The only thing left to specify is the correct cyclic ordering of the d vertices of A_{d-1}. That is understood as follows. Each vertex V of A_{d-1} has two special neighbors, and the graph of the special neighbor relations is a d-cycle. What are the two special neighbors? Well, each vertex of A_{d-1} has a chain of $d-1$ faces (each a $(d-2)$-simplex) containing it, having 60° dihedrals between adjacent members of the chain. Thus each vertex of A_{d-1} regards itself as belonging to exactly two special $(d-2)$-simplex faces – the two ends of the chain. Now every face in this chain shares $d-2$ vertices with each chain-adjacent face, but does not share 1 vertex. Thus the two special chain-end faces each contain 1 special unshared vertex. These two vertices are the two special neighbors of V.

Lemma: If x is the special neighbor of y, then y is the special neighbor of x. **Proof of lemma:** All faces of the A_{d-1} simplex are congruent and have the same set of dihedral angles to other faces – in fact each has exactly two 60° dihedrals to other faces. Further, each face containing a vertex v has either one or two 60° dihedrals to other v-containing faces. So the face (which is an endpoint of x's chain) containing x and y and having exactly one 60° dihedral to another x-containing face f (and f does not contain y) must therefore have exactly one 60° dihedral to another y-containing face, i.e. it is the end of y's chain. Q.E.D.

Another way of looking at the special-neighbor d-cycle within A_{d-1}'s vertices is this: the d faces of A_{d-1} form a d-cycle by traveling from each face to its neighbor via a 60° dihedral. Now each new face we travel to introduces a new vertex, thus defining a d-cycle among the vertices also.

Also note: A_{d-1} is its own mirror image. This causes our simplicial tiling of d-space via tiling A_{d-1}-based prisms not to require mirror image simplices. However, this tiling will in general not be face-to-face because the d-simplices that tile each prism will, in general, do so in a "chiral helical" manner (a "right handed screw"). To make the tiling *face-to-face* we can tile face-adjacent prisms in an oppositely-chiral manner (one left, one right-handed) with opposite chirality d-simplices – so in this case mirror image simplices are employed.

We have thus constructed a 1-parameter family of d-simplices which tile d-space, for each $d \geq 3$.

Alternatives: By stepping along the d-cycle not 1 step at a time, but rather s steps at a time, for any number s relatively prime to d with $2s < d$, we get $\phi(d)/2$ different families of right-handed d-space-tiling d-simplices.

Debrunner [12] explained how to split a $H^d(w)$ simplex into two halves in $\lfloor (d+1)/2 \rfloor$ different ways, thus getting $\lfloor (d+1)/2 \rfloor$ additional families of d-space-tiling d-simplices (for "tilings" consisting of 50% mirror image tiles). He also explained that, if d is even, then there is a way to glue two $H^d(w)$ simplices together to get a new d-space-tiling d-simplex he calls $G^d(w)$. All of these halving and doubling ideas also should work on *all* of our $\phi(d)/2$ different flavors of $H^d(w)$, for a total of $(\lfloor d/2 \rfloor + 2)\phi(d)/2$ different families of d-space-tiling d-simplices – provided "tilings" consisting of 50% mirror image tiles are accepted.

Finally, it is easy to see, for any positive integers k, c, that $c \cdot 2^{(d-1)k}$ copies of our d-simplex is equidecomposable with a correctly chosen d-parallelipiped by simply intersecting the space-tiling with an appropriate d-parallelipiped. Q.E.D.

Explicit coordinates for the vertices of our 1-parameter family of d-space tiling simplices may be gotten as follows. Consider the symmetric circulant $d \times d$ matrix C with first row $(2, -1, 0, 0, \dots, 0, 0, -1)$. Find its $d-1$ mutually orthogonal unit-length eigenvectors – omitting the dth eigenvector, which has zero eigenvalue and is proportional to $(1, 1, \dots, 1, 1)$, Arrange these $d-1$ eigenvectors as the rows of a $(d-1) \times d$ matrix, multiply those rows by the square root of the corresponding eigenvalue of C, and finally take its transpose. The result is d different $(d-1)$-vectors giving the coordinates of the vertices of an origin-centered simplex of type A_{d-1}. Now adjoin an extra (dth) coordinate (the "vertical direction") to each of these vectors. In vector number k, put in this extra coordinate, all integers congruent to sk modulo d. Here s is any fixed integer relatively prime to d with $1 \leq s < d/2$. The result is an infinite number of different d-vectors giving the coordinates of an infinite number of copies of our d-space-filling d-simplex, filling up an infinite vertical simplicial prism. Each copy has as vertices $d+1$ consecutive among these d-vectors, in vertical order. Finally, all the vertical coordinates may be multiplied by an arbitrary nonzero real number, i.e. the free parameter a.

Debrunner also showed how to get numerous particular space-tiling d-simplices by applying halving and doubling operations in various ways starting from Coxeter simplices. However, he got no parameterized *families* in this way, only a long list of specific space-tilers. The Coxeter simplex A_{d-1} is special in that only it has a specially symmetric *d-cycle* of vertices, so only it may be used (apparently) for the prism-based construction above.

Theorem 9. *No regular n-polytope, $n \geq 3$, is dissectable into any other. More generally, for any $n \geq 3$, any n-polytope with only one dihedral angle value θ_1, is not dissectable into any n-polytope with only one dihedral angle value θ_2, if the ratio E_1/E_2 of their total $(n-2)$-flat areas is an irrational algebraic number and the angles θ_1, θ_2 are each CRS angles.*

Proof: The dihedral angles of all the regular n-polytopes (also the non-snub Archimedean solids in 3D), and regular 4-polytopes are CRS angles, see table 1 and [6][10]. Hence they (and their ratios) are transcendental in all cases where they are not trivially rational. It is then easy to see from known formulae (table 1) for the dihedral angles and volumes of the Platonic solids, that no Platonic solid is dissectable into any other. That is because their side-length ratios (at equal volume) are irrational algebraic numbers, due to the volume formulae. Meanwhile, their dihedral angle ratios are either rational or transcendental by theorem 7. Therefore, the product of the two (i.e. the ratio of their Dehn invariants) cannot be rational. Q.E.D.

Open problem: *Does there exist an acute-angled n-simplex that tiles n-space, for any $n \geq 3$?*

Conjecture: *For each $n \geq 2$, any n-polytope is equidecomposable with some n-simplex.*

Open problem: *Does there exist a explicit 2-parameter family of n-simplices tiling n-space, for any $n \geq 3$?*

7 Final remarks

One now may easily show from theorem 4 that $\cos(\pi r)$ is rational for rational r precisely when $r \equiv \{0, 1/3, 1/2, 2/3\}$ mod 1. This was previously known.

Other interesting trigonometric diophantine equations were solved by Coxeter [9], Konematsu & Shirasaka [24], and Conway & Jones [5].

Conway & Jones indeed considered *general* "trigonometric diophantine equations" of the form

$$\text{Rational}\big(x_1, \ldots x_m, r_1, \ldots r_n, \cos(\pi\theta_1), \ldots, \cos(\pi\theta_p)\big) = 0$$

where the θ_k are known rational functions of $x_1, \ldots, x_m$ and $r_1, \ldots, r_n$ and where each x_i is to be integer and each r_j rational. They then showed that any such equation could be converted to an "equivalent" polynomial non-trigonometric diophantine equation, perhaps with extra variables. They showed the original trigonometric diophantine equation, has solutions $(\vec{x}, \vec{r})$ if and only if the new equation has a solution whose non-extra variables agree with $(\vec{x}, \vec{r})$. This constitutes a general purpose method for attacking such problems. However, the Conway-Jones "reduction" does not necessarily represent forward progress since solving polynomial diophantine equations was shown by Matiyasevich and Robinson [27] to be a Turing-undecidable problem, and since sometimes the original trigonometric formulation seems easier to handle.

Conway & Jones wrongly claimed that the problem of finding all tetrahedra equidecomposable with a cube was a trigonometric diophantine equation of their sort and hence equivalent to an ordinary polynomial diophantine equation. This claim is false, or at least highly misleading, for two different reasons (indeed it is false in dimensions $n = 2$, 3, and 4). First, the set of n-simplices equidecomposable with the cube is, as we have seen, *continuum* infinite, which is not "diophantine" behavior at all.

Second, in 3D and 4D I do not believe the problem can be cast in Conway & Jones' form, i.e. without using forbidden functions such as inverse-trig or square roots. (In 2D it can, but there the "trigonometric diophantine equation" is just $\theta_1 + \theta_2 + \theta_3 = \pi$, which is hardly of interest and there is no rationality requirement imposed on the θ_j.)

We close with

Theorem 10. *The third side of a singly-right spherical triangle with two rational sides (measured in degrees) must be transcendental (except in the all-rational cases, for which see theorem 2).*

Proof. The fact that it is either rational or transcendental is an immediate consequence of the Gelfond-Schneider-Baker theorem. Q.E.D.

References

[1] M. Abramowitz & I.A. Stegun: Handbook of Mathematical Functions, 9th Edition. Dover, New York 1972.

[2] A.Baker: Transcendental Number theory, Cambridge Univ. Press 1975.

[3] V.G.Boltyanskii: Hilbert's third problem, V.H.Winston & Sons, Washington DC 1978.

[4] J.M.Borwein & R.Girgensohn: Addition theorems and binary expansions, Canadian J. Math. 47,2 (1995) 262-273.

[5] J.H. Conway & A. Jones: Trigonometric diophantine equations (on vanishing sums of roots of unity) Acta Arithmetica 30,3 (1976) 229-240.

[6] J.H.Conway, C.Radin, L.Sadun: On angles whose squared trigonometric functions are rational, Discr.Comput.Geom. 22 (1999) 321-332.

[7] J.H. Conway and N.J.A. Sloane: Sphere Packings, Lattices, and Groups, Springer-Verlag, second edition, 1993.

[8] H.S.M. Coxeter: Discrete groups generated by reflections, Annals of Math. 35 (1934) 588-621.

[9] H.S.M. Coxeter: Problem 4136, Amer. Math'l Monthly 53,2 (Dec 1946) 588.

[10] H.S.M. Coxeter: Regular polytopes, Dover (3rd ed.) 1973.

[11] H.S.M. Coxeter: NonEuclidean geometry, MAA (6th ed.) 1996.

[12] Hans E. Debrunner: Tiling Euclidean d-space with congruent simplices, Discrete geometry and convexity, Proc. Conf., New York 1982, published as Annals N.Y. Acad. Sci. 440 (1985) 230-261.

[13] E.J.Eckert: The group of primitive Pythagorean triples, Math. Magazine 57 (1984) 22-27.

[14] W. Fenchel: Elementary Geometry in Hyperbolic Space, Walter de Gruyter, Berlin 1989.

[15] M.Goldberg: 3 Infinite families of tetrahedral space fillers, J. Combinatorial Theory A 16 (1974) 348-354.

[16] L. C. Grove & C. T. Benson: Finite Reflection Groups, Springer Verlag (GTM #99) 2nd ed. 1996.

[17] Hugo Hadwiger: Hillsche Hypertetraeder, Gaz. Math. (Lisbon) 12,50 (1951) 47-48.

[18] H.Hancock: Foundations of the Theory of Algebraic Numbers, 2 Vols., Macmillan, New York 1931/1932.

[19] G.H.Hardy & E.Wright: An Introduction to the Theory of Numbers, 5th ed. Oxford University Press 1985.

[20] M.J.M. Hill: Determination of the volumes of certain species of tetrahedra, Proc. London Math. Soc. 27 (1896) 39-53.

[21] C. D. Hodgson, I. Rivin, and W. D. Smith; A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere. Bull. (New Series) of the AMS, 27,2 (1992) 246-251.

[22] J. E. Humphreys: Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics #29, Cambridge University Press 1990.

[23] H.Koch: Number Theory: Algebraic Numbers and Functions, Amer. Math. Soc., 2000.

[24] Shigeru Konematsu & Shigeshi Shirasaka: see Math'l Reviews MR90g:11042.

[25] Serge Lang: Introduction to transcendental numbers, Addison-Wesley 1966.

[26] Barbara H. Margolius: Plouffe's constant is transcendental, `http://www.lacim.uqam.ca/~plouffe/articles/plouffe.pdf`.

[27] Yuri Matiyasevich: Hilbert's tenth problem, The MIT Press, Cambridge MA, 1993.

[28] I.Niven: Irrational numbers, Math'l Assoc. America (Carus Monograph #11) 1956.

[29] L.W. Reid: Elements of the Theory of Algebraic Numbers, Johns Hopkins University Press, Baltimore MD 1946. Now available from University Microfilms "Books on Demand."

[30] M.Senechal: Which tetrahedra fill space, Mathematics Magazine 54,5 (1981) 227-243.

[31] D.Shanks: Solved and Unsolved Problems in Number Theory, 4th ed. Chelsea, New York 1993.

[32] A.B.Shidlovskii: Transcendental Numbers, de Gruyter, New York 1989.

[33] Darko Veljan: 2500 years of Pythagoras's theorem, Mathematics Magazine 73,4 (2000) 259-272.

[34] J.Volder: The CORDIC trigonometric computing technique, IRE transactions electronic computing EC-8 (1959) 330-334.

[35] J.S.Walther: A unified algorithm for elementary functions, Spring joint computer conference (1971) 379-385.

[36] A.Wiles: Modular Elliptic-Curves and Fermat's Last Theorem, Annals Math. 141 (1995) 443-551.