The computational power of iterating analytic functions

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Abstract —

We describe a meromorphic function $g(z)$ and an entire analytic function $h(z)$ whose iterations each simulate a universal computer. The simulation is unaffected by "noise" of bounded magnitude. Consequently, numerous problems about such iterations are undecidable.

There are senses in which $g(z)$ and $h(z)$ are simplest possible: we show iterating rational functions cannot cause Turing universal behavior.

Finally, we consider Blum, Cucker, Shub, and Smale's notion of "undecidable" problems for a machine capable of performing real arithmetic operations. B,C,S&S had claimed without proof that membership of a point in the "Mandelbrot set" M was undecidable in their sense. Although we provide a proof, we argue that this statement is very deceptive. For example, we show (assuming Douady's conjecture that M is "locally connected") that this problem *is* decidable if the point is known to be either in the interior or complement of M, or if the point is known to have algebraic coordinates. Neither the inscrutable nature of M, nor the fact that its boundary has Haussdorf dimension 2, need have anything to do with the undecidability. To see that, we show the well behaved set $S_e = \{x, y | y \le e^x\}$ is BCSS-undecidable. Again, though, we have decidability if (x, y) is known to lie in the interior or complement of S_e, or if x and y are known to be algebraic numbers. For these reasons we recommend (& provide) a redefinition of BCSS-undecidability.

Keywords — Elliptic functions, meromorphic functions, simple computers, counter machines, Turing universal, entire functions, iteration of rational functions, repetition count functions, Mandelbrot set, Julia set.

1 Introduction

1.1 A meromorphic function which is a universal computer

60 years ago it ceased to be a surprise that iterating a function can give you Turing universal computational power.

But we will show in §2 that even if that function has to be a *meromorphic*[1] function of just *one* complex variable (made entirely of "standard" functions described in [22]) then it can still have Turing universal power, and even in the presence of (bounded magnitude, malicious) noise.

Our function $g(z)$ has an infinite number of poles and zeros, and (consequently) has an essential singularity at ∞. One therefore is motivated to ask: can a meromorphic function without an essential singularity at ∞, i.e. a rational function, exhibit noise-immune Turing universal iteration behavior? In §3 we show that the answer is "no."

Thus in some sense the function $g(z)$ we propose is simplest possible.

We'll also see in §4 that there is an "entire" [11] analytic (i.e. without any singularities, not even poles, in the finite complex plane) function $h(z)$ whose iteration simulates a Turing universal computer. This simulation is also noise-immune.

1.2 Machines that perform real arithmetic

Blum, Cucker, Shub, and Smale in their book [10] develop the theory of the computational power of machines that perform real arithmetic operations. In particular they introduced a notion of "BCSS-undecidable" computational yes/no problem – those insoluble by their machines. Their prime example of such a problem (although no proof of this was provided in [10]) was membership in the "Mandelbrot set" M.

In §5 we will provide a proof. But more importantly, we will argue that the whole BCSS-undecidability notion is wrongheaded and essentially uninteresting and useless. This is dramatized by the fact that membership in the set of (x, y) in $\mathbf{R}^2$ such that $y \le e^x$ is BCSS-undecidable! We propose (definition 8) a modified definition of BCSS-undecidability designed to regain interest. All the examples of BCSS-undecidable sets proposed by B,C,S&S [10] are decidable according to the modified definition (although I must qualify this: to show the decidability

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[1] That is, analytic, i.e. infinitely complex-differentiable, except possibly for poles. In particular there are no "branch points." Indeed if viewed as a map from the complex numbers $\mathbf{C}$ to the Riemann *sphere* $\mathbf{C} \cup \infty$, meromorphic maps are analytic everywhere.

of Mandelbrot membership, I needed to assume conjecture 12, due to Douady, that M is "locally connected"). But, we show in theorem 9, genuinely undecidable sets do exist, with the modified undecidability definition.

2 Turing Universal Meromorphic function

2.1 Simple constructions of computers

There are several especially simple constructions of Turing universal computers. Minsky [26] (and see our §2.2, 2.3) describes the "counter machine." A modification of this by Conway called "FRACTRAN" (see our §4) involves only a single large integer serving as the machine's "memory." A very simple 2-dimensional 2-state cellular automaton called "life" supports universal computers [6]. (The universality proof sketched in [6] will only suffice to emulate N steps of a universal Turing machine in $\theta(2^N)$ steps – rather inefficient. This can be improved to $\theta(N)$ by using an "extensible delay line memory" devised by Paul Callahan [14].) Matthew Cook [15] recently showed in unpublished work that a very simple 1-dimensional 2-state cellular automaton, called by him and Conway "LeftLife," is Turing universal. Its transition rule is

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before 000 001 010 011 100 101 110 111
after   0   1   1   1   0   1   1   0
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LeftLife also leads to the simplest known universal Turing machines: For 2,3,4,5 symbols, 7,4,3,2 states respectively suffice[2].

2.2 Strategy

Our function $g(z)$ has the following form.

$$g(z) = \mathrm{Rational}\big(\mathrm{Elliptic}(z), \tanh(Kz), \tan(Kz)\big) \quad (1)$$

Different elements of this construction will be described in different subsections of the rest of this section. The overall strategy is as follows. It was shown by Minsky [26] (and see §2.3) that a "2-counter machine" suffices for Turing universal power.

A k-counter machine has k "counters" (nonnegative integers) and a finite "program" consisting of two kinds of instructions:

- *Increment*(j)*:* Increment counter j, $j \in \{1, \ldots, k\}$.
- *Dec&Test*(j,n)*:* If counter j contains zero, go to instruction n, otherwise decrement it.

The function in EQ (1) is designed to emulate a 2-counter machine. We use an elliptic function whose two periods are 1 and i; the two counters are $\lfloor \mathrm{Re}(z) \rfloor$ and $\lfloor \mathrm{Im}(z) \rfloor$. The rational function describes the "state transition table," i.e., the finite "program." It also serves to self-correct z to eliminate noise and errors. The role of the tanh and tan functions is to detect zero counter values. K is some integer which needs to be made sufficiently large (but finite) to make everything work; $K = 2^{10}$ should suffice.

2.3 The universality of 2-counter machines

Theorem 1 (Minsky's [26] universality theorem for 2-counter machines) *There is a certain specific finite "universal interpreter" program for the 2-counter machine, which can be made, by choice of the initial data in the counters, to emulate any program on any Turing machine with any input data.*

Proof sketch. First, it is easy to program a 3-counter machine to emulate a Turing machine step on an N-long tape in $2^{O(N)}$ steps. (One of the counters, which is an integer of order $2^{O(N)}$, stores the N-symbol Turing machine tape, the other two are used to count to 2^k in order to change the kth tape symbol, which is where the Turing machine's "read/write head" is located.) Second, Minsky shows how to program a 2-counter machine to emulate a single step by a 3-counter machine (where the contents of the three counters are the integers A, B, and C) in $O(2^A 3^B 5^C)$ steps. (The three counters A, B, C are stored in the single integer $2^A 3^B 5^C$, and increments and decrements are accomplished by multiplying and dividing this integer by 2,3, or 5, which is accomplished by counting the free counter 2,3, or 5 times faster than the storage counter.) The combined result is that a 2-counter machine running for $30^{2^{O(N)}}$ steps can simulate a single step of any Turing machine with an N-symbol long tape. ☐

2.4 The elliptic function

"Elliptic functions" are meromorphic, doubly periodic functions. For concreteness we will demand that our elliptic function have periods 1 and i, and then the unique (up to algebraic transformations) elliptic function is $\mathrm{sl}(z)$, the functional inverse to the "lemniscate arc length integral"

$$\frac{1}{4K} \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^4}} \quad (2)$$

where

$$K = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} \approx 1.311028778. \quad (3)$$

A large number of beautiful properties of sl are listed in the appendix.

2.5 The rational function

Imagine that our 2-counter machine consists of two counters (each a nonnegative integer) plus a finite set of

[2] Differences from previous-generation UTMs: (1) The tape needs to start out with a periodic repeating pattern, rather than blank. (2) The machines do not halt. Some people find one or the other of these upsetting. Cook's emulation of the universal Turing machine suffers exponential slowdown; this might be improvable to polynomial slowdown with additional work. I sent Cook a sketchy proof that LeftLife is the simplest Turing universal 1D cellular automaton.

"states." The two counters may be thought of as describing which square cell of the infinite periodic lattice, with periods 1 and i, z is in. Within each such cell, there are a finite number of points $p_1, p_2, \ldots$, and which "state" we are in is described by which of these points we are in the neighborhood of. Specifically, we are in state n if $|(z \bmod [1,i]) - p_n| < \epsilon$ where ϵ is some sufficiently small constant. (If S is the number of states, then $\epsilon = 0.1/\sqrt{S}$ should suffice.)

The rational function is constructed, by means of a rational interpolation, to map $\text{Elliptic}(p_j) \to p_k$ when the two other (tan and tanh) arguments are 1 and where the state transition table says to transition from state j to state k.

Note, by intentionally moving to $p_k \pm 1 \pm i$ rather than just p_k, the rational function can cause counter increments and decrements (indeed by any fixed set of constant 2D step sizes) to be performed with each transition.

By constructing the rational function correctly (for example, by composing it with a Newton iteration[3] converging to the roots of the polynomial $\prod_j (z - p_j)$) so that the p_j are *attracting* points, we can make some bounded amount of noise (added to f at every step) irrelevant.

2.6 The zero detectors

Note that $\tanh(Kz)$ is exponentially close to ± 1 if z is not pure-imaginary and if K is made sufficiently large. Indeed, if z is at least Δ away from the imaginary axis, then $|\tanh(Kz) - \text{sign}(\text{Re}\, z)| < \text{sech}(-K\Delta)^2$. (This may be seen after first realizing, e.g., that $w = e^z$ maps the left halfplane $\text{Im} z < 0$ to the unit disc $|w| < 1$.) Hence $\tanh(Kz)$ serves as a zero detector (actually, a sign indicator) for the real-part counter.

$\text{Tan}(Kz)$ plays the analogous role for the imaginary-part counter.

Our machine would be perfect if these tan and tanh terms were exactly ± 1 according to the sign of the two counters, but unfortunately there are exponentially small errors if K is large but not infinite. However, due to the fact that error correction may be built into the rational function (as we've already said) for large enough K this won't matter. It should suffice if $K > 10\sqrt{S}\log S$ where S is the number of states.

2.7 Conclusion

Consider the iteration

$$z_0 = z + \eta_0, \quad z_n = g(z_{n-1}) + \eta_n, \ \ n \ge 1 \qquad (4)$$

where the "noise" terms η_j are arbitrary complex numbers obeying $|\eta_j| < \epsilon$, where ϵ is a sufficiently small real constant. It should suffice if $\epsilon < 0.1/\sqrt{S}$ where S is the number of states.

[3] Actually, all that is required to make the p_j attractive is that $|f'(p_j)|$ is bounded below 1. The Newton idea may be used to get "super attraction," i.e. $f'(p_j) = 0$.

Theorem 2 (Turing universal meromorphic functions) *Meromorphic functions $g(z)$ of the form EQ (1) as described above, exist, whose iteration, even in the presence of malicious noise as in EQ (4), will simulate a (noiseless) universal Turing machine whose tape contents determine and are determined by (by a simple algorithmic mapping) $\lfloor \text{Re}\, z \rfloor$ and $\lfloor \text{Im}\, z \rfloor$.*

Consequently, essentially all questions about the long term behavior of the sequence of iterates of $g(z)$ are Turing-undecidable, including whether any starting point has iterates that diverge to ∞, or whether the sequence of iterates arising from some starting point ever enters some disc. □

Remark. Since our $g(z)$ achieves Turing universality without ever leaving the quadrant $\text{Re}\, z, \text{Im}\, z > 0$ (well, more precisely, $z + i + 1$ never leaves this quadrant), one may feel that the other 3 quadrants are "wasted." This waste could be eliminated by using $G(z) = [g(z^{1/4} - 1 - i) + 1 + i]^4$ instead of $g(z)$. This would also have the benefit of enhanced noise immunity (the magnitude of the noise is now allowed to grow proportionally to $|z|^3$). But, this $G(z)$ is no longer meromorphic.

Remark. We can also obtain immunity to noise in the *definition* of $g(z)$, i.e. the Turing machine would still work if all the poles and zeros of $g(z)$ were independently perturbed by some sufficiently small universally bounded amounts, or if g were perturbed in the Haussdorf topology on the Riemann sphere by a sufficiently small bounded amount.

3 Simpler functions cannot be Turing universal

Theorem 3 (Rational function iteration with noise) *Rational function iteration cannot be Turing Universal in the presence of arbitrarily small (but nonzero) noise.*

Proof. Before we begin, we'll need to discuss some of the terminology and known results [5] concerning iterations of rational functions $R(z)$. Let $R^{[n]}(z)$ denote the n-time iterate of R at z. Let $\bar{\mathbf{C}}$ denote the extended complex plane $\mathbf{C} \cup \infty$. The set of $z \in \bar{\mathbf{C}}$ such that $R^{[n]}(z)$ is equicontinuous in the Riemann sphere metric for all $n \ge 0$, is called the "Fatou set of R." The "Julia set of R" is the complement in $\bar{\mathbf{C}}$ of the Fatou set.

Theorem 4.2.3 p.68 of [5] shows that the Julia set of a rational function is either (1) the entirety of $\bar{\mathbf{C}}$, or (2) has no interior.

Theorem 4.3.2 p.76 of [5] shows that case (1) happens if and only if there exists some z such that the forward iterates of that z are dense everywhere in $\bar{\mathbf{C}}$. In that case, Turing universality in the presence of noise is obviously impossible since the noise could perturb all z (i.e. *any* Turing machine state) into the same future sequence.

Hence WLOG we are in case (2) – the Julia set has zero interior – and due to the noise we may assume WLOG that we are in the Fatou set.

It is known (e.g. [5] theorem 5.5.4 p.92) that each Fatou set connected[4] component (FSCC) is mapped bodily homeomorphically into another FSCC (possibly an m-fold map for some integer m, and possibly this is a self-map).

Now in order for our Turing machine to work in the presence of noise, each "state" must correspond to some open subset of $\bar{\mathbf{C}}$. Presumably the map, within these subsets, has some kind of contractive, attractive, behavior to cause noise immunity. Call two states "equivalent" if they have the same future symbolic dynamics (i.e. the same future sequence of logical states). Otherwise they are "inequivalent."

We claim that it is impossible for two inequivalent states to lie inside the same FSCC. (Suppose not. Then there is some continuous path lying entirely within that FSCC connecting elements of one state to elements of another. At some point z as we walk along this path, the future symbolic dynamics must change instantaneously. That means that either we were not noise-immune, or [if the corrective behavior pulled you away from the danger region by a distance bounded below by a constant, so that we are noise-immune] some $R^{[n]}(z)$ is not equicontinuous at a z inside an FSCC – a contradiction.) Hence, each FSCC corresponds to ≤ 1 logical state.

Now, Sullivan's "no wandering domains" theorem ([5] chapter 8)[5] shows that every state sequence is ultimately periodic. This prevents Turing universality because Turing machines are capable of never halting while never entering a state-cycle. □

4 A faster computer from iteration of an entire function based on FRACTRAN

J.H.Conway's "FRACTRAN" (in one incarnation) is a Turing-universal functional iteration on the natural numbers of the form

$$F(n) \;=\; A_{n \bmod M} \;+\; n\, B_{n \bmod M}. \tag{5}$$

The computer is described by its modulus M (some finite number) and two tables of M integer values $A_0, A_1, \ldots, A_{M-1}$ and M rational values $B_0, B_1, \ldots, B_{M-1}$, where the denominators of the B's must be chosen in such a way that $nB_{n \bmod M}$ will always be an integer if n is.

The fact that such a machine can have Turing universal power is an easy consequence of the fact (theorem 1) that a 3-counter machine can. The FRACTRAN machine simulates a 3-counter machine by storing all three counters A, B, C in a single integer $n = 2^A 3^B 5^C$ and increments and decrements the counters by multiplying or dividing by 2,3, or 5. M is some number of the form $2^a 3^b 5^c$ for some sufficiently large constants a, b, c.

It is possible to turn this into an iteration on the real or complex numbers

$$F(z) \;=\; \mathrm{trigpoly}_1(z) + z\, \mathrm{trigpoly}_2(z) \tag{6}$$

which happens to degenerate (in the case when z is an integer) to the original FRACTRAN function. Also, it is possible by use of the fact that, e.g.,

$$z \to z - \frac{\sin(2\pi z)}{2\pi} \tag{7}$$

has superattractive fixed points at integer z, even to make the Turing universality of this F *stable* to infinitesimal perturbations.

Unfortunately, the machine represented by this F still is destroyed by any *finite* amount of noise, no matter how small its upper bound. It is possible to overcome that problem by using a better "integerizing corrector" than EQ (7). Specifically, consider the map $z \to C(z)$ where

$$C(z) = z - \frac{\sin(2\pi z)}{2\pi} \sum_{n \geq 0} a_{2n} \tanh\left(\frac{e^z}{n!}\right) \sin(2\pi z)^{2n} \tag{8}$$

where a_{2n} are the coefficients of t^{2n} in the series (convergent for $|t| < 1$)

$$\frac{\arcsin t}{t} = 1 + \frac{t^2}{2 \cdot 3} + \frac{1 \cdot 3 t^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 t^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots \tag{9}$$

$C(z)$ is an entire analytic function (since the series EQ 8 converges for all z and so does the series for $C'(z)$) with attractive fixed points at every positive integer z. It maps complex numbers z which are within a constant (say 0.1) of being a positive integer, to within $< 0.1/|z|$ of that integer. (This is because $\gg \log x$, in fact $> \sqrt{x}$, where $x = \mathrm{Re} z$, terms of the arcsin series occur before the tanh term cuts them off. Therefore the sum of the cut-off terms in the series is $O(1/x)$. Furthermore, although the "cutoff" induced by the tanh is not sharp, the individual errors this induces in the first $\log x$ terms of that series are each exponentially small for x large.) *This* kind of self-corrective behavior (as opposed to the weaker behavior of EQ 7) is good enough (with appropriately changed constants) to cause $h(z) = F(C(z))$, with the F from EQ (6), to be immune to noise of absolute value bounded by 0.1.

Theorem 4 (Turing universal Entire function) *There exists an entire function $h(z)$ whose iteration, even in the presence of malicious noise whose absolute value is bounded by 0.1, will simulate N steps of a universal Turing machine in $2^{O(N)}$ iterations.*

Remarks.

1. This Turing machine emulation is faster than our simulation in theorem 2.

[4] A set is "connected" if it is not contained in the union of two disjoint open sets. The "Jordan curve theorem" states that a set $S \subset \mathbf{R}^2$ is disconnected if and only if there exists a simple closed curve separating S.

[5] If for integers n, m, $0 \leq m < n$, $R^{[m]}(\Omega) = R^{[n]}(\Omega)$ then Ω is said to be "eventually periodic." On the other hand, if all the $R^{[n]}(\Omega)$, $n \geq 0$, are disjoint, then Ω is a "wandering domain." Sullivan's theorem states that if Ω is a connected component of a Fatou set of a rational function R, then Ω is eventually periodic.

2. If z is restricted to the real axis, h-iteration still is Turing universal and noise immune (even for complex-valued noise).

3. This result also, incidentally, represents a new, and far simpler, proof of the fact [2][3][18] that entire functions, unlike rational functions, can have "wandering domains."

5 The Mandelbrot set and BCSS-undecidability

Definition 5 *The "Mandelbrot set" M (actually due to Brooks and Matelski [13]) is the set of all complex numbers c such that all iterates of $|f_c(0)|$, $|f_c(f_c(0))|$, $|f_c(f_c(f_c(0)))|$ are bounded, where $f_c(z) = z^2 + c$.*

It is easily seen [5] that this sequence diverges to ∞ if and only if some iterate z obeys $|z| > 2$.

Blum, Cucker, Shub, and Smale [10] (from here on, we shall refer to them as "BCSS") consider a "computer" which can do any of the the following in 1 step: perform a real arithmetic operation, store, copy, input, or output a real number, or branch based on the result of a comparison test ($a < b$). They define a set S of real D-tuples to be "decidable" in their model of computation if there is some program for their computer, which, provided with $\vec{x} \in \mathbf{R}^D$, will eventually always output "yes" or "no" if $\vec{x} \in S$ or $\vec{x} \notin S$ respectively. A necessarily condition for S to be BCSS-decidable is that both S and its complement $\overline{S}$ be countable unions of semi-algebraic sets (CUASs).

They then state without proof ([10] section 2.4) that M is an undecidable set when $D = 2$ because it is not a CUAS (we'll prove this in theorem 15), and that this in turn is somehow due to the fact that M "has Haussdorf dimension 2."

Furthermore they claim (also without proof) that any set with non-integral Haussdorf dimension is BCSS-undecidable, and that most (and many particular) "Julia sets" also are BCSS-undecidable.

An explicit example of a BCSS-undecidable set S_{Pn} is as follows (based on a proof sketch provided by BCSS [10]):

Theorem 6 *For any real polynomial $P(x)$ with all roots real and with at least 4 distinct real roots, let $S_{Pn} \subset \mathbf{R}$ be the set of real x from which Newton's method does not converge to a root of $P(x)$. Then: S_{Pn} is BCSS-undecidable.*

Proof. Barna [4] proved (though see [34] for the slightly improved result we've used) that for such polynomials, Newton's method converges to a root from a full-measure set of starting points, and the exceptional points form a measure-0 set homeomorphic to a "Cantor set" (with uncountably infinite cardinality, but nowhere dense). Hence S_{Pn} cannot be CUAS. □

A simpler Cantor set is the set of all real x with $0 < x < 1$ whose ternary representation has no "1" digits.

In my opinion, the whole notion of BCSS-undecidability is overhyped, surrounded by an unnecessary aura of mystery, and rather useless and uninteresting. To support this opinion

1. I'll prove in theorem 16 that entirely un-mysterious sets in $\mathbf{R}^2$ such as $S_e = \{x, y \,|\, y \le e^x\}$ are BCSS-undecidable. (I state without proof: the set $S_\Gamma = \{x, y \,|\, y \le \Gamma(x)\}$ also is.) Surely, any notion of "undecidability" that includes such a set is not very interesting?

2. Both S_e and S_Γ, and the Mandelbrot set M (under Douady's conjecture 12 that it is "locally connected"), *are* decidable if it is known in advance that x and y are algebraic numbers. Furthermore, all these, and all Julia sets, and all Cantor sets, *are* decidable if it is known that (x, y) is either in the interior or complement of the set (i.e. the only difficulty arises for points exactly on the boundary). See theorems 10, 13, 14.

I think, therefore, that it would be more productive to focus, not on BCSS-undecidability, but rather on a redefined notion of it.

One attempt at a redefinition is as follows.

Definition 7 (Algebraic BCSS-undecidability) *Same as BCSS undecidability, but only algebraic numbers, rather than all real numbers, are considered.*

Since a BCSS-machine can only generate rational numbers if it is started with $\{0, 1\}$ as the only available constants; and it can only generate algebraic numbers if the operation set is expanded to allow solution of polynomial equations; this seems to be a useful concept – restrict attention to *accessible* (and checkable as such) reals. But algebraic BCSS-undecidability is then merely *exactly the same* as conventional Turing undecidability, since a Turing machine can simulate a BCSS machine restricted to algebraic numbers, and vice versa. A more interesting redefinition is:

Definition 8 (BCSS-undecidability with noise) *Same as BCSS undecidability, except that the decision procedure is allowed to return the wrong answer for $\vec{x}$, if that wrong answer is actually correct for input points $\vec{y}$ unboundedly near $\vec{x}$. In other words, we only require the decision procedure to discriminate between points $\vec{x}$ in the interior or complement of our set; for $\vec{x}$ on its boundary either answer is acceptable.*

In either of these two redefinitions, S_e and M (under conjecture 12) are both decidable, and under the second, Julia sets and Cantor sets[6] become decidable too.

Since all BCSS's examples of undecidable sets become decidable under modified definition 8 (at least, under the Douady conjecture 12) one then asks: are there any easy examples of undecidable sets (with these redefinitions)? The answer is yes.

[6]Proof: Since Cantor sets consist of isolated points, we are allowed always to return "no."

Theorem 9 (Undecidable sets for redefined BCSS undecidability) *Consider the set S_g of complex numbers z such that the sequence $g(z)$, $g(g(z))$, $g(g(g(z)))...$, remains bounded, where g is one of our two previously defined types of computation-universal meromorphic (or entire analytic) functions from theorems 2, 4. This set is "undecidable" with either of the two redefinitions of BCSS undecidability.*

Proof. Assume (for a contradiction) there were a BCSS decision procedure for such sets. Then:

1. In the all-algebraic number scenario, a conventional Turing machine could, by performing exact arithmetic on algebraic numbers, emulate the BCSS machine, so we'd have a conventional decision procedure. But obviously S_g (considered within the universe of points with algebraic coordinates) is Turing undecidable since the "Halting problem" for Turing machines is.

2. Consider the finite sequence F of exact real arithmetic operations and logical state transitions performed by the BCSS machine on some input $\vec{x}$ and terminating with a yes/no answer. That sequence occurs precisely for $\vec{x}$ in some particular finite union S_F of semi-algebraic sets. By appropriate choice of the unboundedly small "noise" plaguing the BCSS machine, we may force it always to regard $\vec{x}$ as not lying on ∂S_F for any F. (For example, make the coordinates of $\vec{x}$ be algebraically independent transcendentals.)

But membership in the interior or exterior of S_F is decidable by a conventional Turing machine using "interval arithmetic" with escalating precision. (We assume the Turing machine is provided with an infinite input tape containing all decimals of $\vec{x}$.) Hence, a Turing machine could be devised which would always halt (if the BCSS machine halted) and would then always return the same answer as the BCSS machine, *except* for at $\vec{x} \in \cup_F \partial S_F$, in which case, the Turing machine might never halt. This is a countable union of algebraic sets of measure zero, hence has measure zero.

So in particular, a Turing machine would exist which would halt with the right answer for S_g membership at a full measure set of (x, y), and would never halt with the wrong answer, but for a zero measure set S_Z could run forever. Furthermore, the bad set S_Z would consist entirely of points with algebraically related coordinates. Now a Turing machine can intentionally preperturb $\vec{x} = (x, y)$ to $\vec{x}' = (x + e^{-c}, y + e^{-c\sqrt{2}})$ or $\vec{x}'' = (x + e^{-c\sqrt{3}}, y + e^{-c\sqrt{5}})$ for some large positive integer c. All numbers $e^{-\sqrt{j}}$ are known to be algebraically independent (c.f. the proof of theorem 16). Hence, it is impossible for there to be some algebraic relation between x and y inside all three[7] of $\{\vec{x}, \vec{x}', \vec{x}''\}$. Hence, our Turing machine would be guaranteed always to halt on (and hence decide membership of) at least one of $\vec{x}$ or $\vec{x}'$ or $\vec{x}''$. But it it clear (by choosing c small enough) that even *this* modified membership problem is Turing undecidable since it would permit solving the Halting problem.

In either case we get a contradiction. □

Theorem 10 (Julia sets) *Julia sets of rational functions $R(z)$ are decidable according to definition 8.*

Proof. As we've already said in the proof of theorem 3, Julia sets of rational functions either (1) have empty interior (example: $R(z) = z^2 - 1/2$) in which case, we are allowed always to return "no," or (2) are the entire complex plane (example; $R(z) = (z-2)^2/z^2$) in which case we must always return "yes."

Thus to decide membership in the Julia set of any *particular* R, a trivial decision procedure suffices. New problem variant: If we want an algorithm which *inputs* the (complex number) coefficients of R, as opposed to regarding R as fixed, then we are allowed always to output "no" since case 2 occurs on a set of $R(z)$ of measure 0. This is because in case 2 it is known (Theorem 4.3.2 p.76 of [5]) that some point must exist whose forward iterates are everywhere dense. Hence we can disprove case 2 by exhibiting an attractive cycle or repulsive cycle. But, since a finite nonzero number of fixpoints (solutions of $R(z) = z$) always exist by the fundamental theorem of algebra, and such a z is repulsive if $|R'(z)| > 1$ and attractive if $|R'(z)| < 1$, we see that case 1 can only occur when $|R'(z)| = 1$, which is a measure-0 event. □

Definition 11 (Local connectedness) *A set S is said to be "locally connected" if any piece $U \cap S$ of S ($S \subset \mathbf{C}$ open) has the property that for any $z \in U \cap S$, there is a neighborhood $V \subset U$, $z \in V$, such that $V \cap S$ is connected.*

Conjecture 12 (Douady) *The Mandelbrot set M is locally connected.*

Remark. Douady and Hubbard [17][5] proved M is simply connected by exhibiting a conformal mapping between ext$M$ and $|z| > 1$.

Douady [27] showed that conjecture 12 had numerous consequences:

1. A theorem of C.Caratheodory says that the conformal map has a continuous extension to ∂M if and only if M is locally connected.

2. Conjecture 12 implies that the points $c \in \text{int}M$ are exactly the c such that f_c has an attracting cycle.

Theorem 13 (Interior of Mandelbrot set) *Conjecture 12 implies: there exists a BCSS-algorithm which, given (x, y), will halt with a proof $(x, y) \notin M$, or halt with a proof $(x, y) \in M$, unless (x, y) is in $M \cap \partial M$ or in a set of measure 0.*

Proof. The algorithm is as follows. Start with $z = 0$ and keep on iterating $z \leftarrow f_c(z)$. (See definition 5.) Then

[7] A D-dimensional version of this argument, were we interested in extending it beyond just $D = 2$, could be based on $2^D - 1$ versions of $\vec{x}$.

WLOG z will eventually get sucked toward some attracting cycle. (We'll prove that soon.) Postulate the period of the cycle is Q and solve for the fixpoints of $f_c^{[Q]}$ near to z (this is finding the roots of a polynomial and is algorithmic) and prove z is within the trivial attracting basin (that is, the region where $|f_c^{[Q]'}(z)| < 1$ containing that fixpoint) of such a fixpoint. (This is algorithmic since it is a polyomial inequality.) If no trivial proof exists, continue on with more iterations of f_c and more postulations about Q. (On the nth iteration, try all possible postulations $Q = 1, 2, \ldots, n$.) Since we'll be making the right postulate about Q an infinite number of times and we keep getting sucked closer to the fixpoint, eventually z will be in the trivially provable attractive basin of that fixpoint and the algorithm will terminate with a proof $c \in \mathrm{int}M$.

This algorithm will also produce proofs c is in the *exterior* of M if we view ∞ as another "fixpoint" of period 1. Indeed [5], $c \notin M$ iff some iterate $z = f_c(0), f_c(f_c(0))$, etc. has $|z| > 2$. In another variant of the algorithm, we do not need to prove any z is within an attracting basin of a fixpoint; it suffices merely to prove that an attractive fixpoint *exists*, i.e. that some complex y, $y \neq \infty$ exists such that $f_c^{[Q]}(y) = y$ and $|f_c^{[Q]'}(y)| < 1$. Answering this question is of course algorithmically feasible (it is membership in a semialgebraic set). In practice such a fixpoint could be found by a subsidiary Newton iteration to converge rapidly to a root of the polynomial $f_c^{[Q]}(z) - z$; one would use the current iterate of $z = f_c(0), f_c(f_c(0)), \ldots$ as Newton's starting point.

This algorithm can only fail to terminate if c is in M and

1. $z = 0$ is not in the basin of an attractive cycle or
2. $f_c(z) = z^2 + c$ has no attractive cycle, stable cycle, or Siegel ring, but c is nevertheless in M.

Under the Douady conjecture, $z^2 + c$ can fail to have an attractive cycle (for $c \in M$) only if $c \in \partial M$, which is excluded by assumption. Hence case 2 is impossible. (See the 5 possibilities in [5] chapter 7.)

Then case 1 is known to be impossible because there are only two "critical points" (0 and ∞) of $z^2 + c$, and it is a theorem of P.Fatou [5] that each attractive cycle must contain at least one critical point in its attractive basin. □

Theorem 14 (Mandelbrot set for algebraic points) *Under conjecture 12: If x and y are known to be algebraic numbers, then there is a decision procedure for determining whether $x + iy$ is in M.*

Proof. The algorithm in the previous proof handles $c = x + iy$ such that $c \in \mathrm{int}M$ or $c \notin M$. We'll fill in the remaining gap by showing that if x and y are algebraic and $c \in M \cap \partial M$, then (1) this situation is algorithmically recognizable, and (2) $x + iy \in M$. Under the Douady conjecture [27], the only points in M which the previous algorithm could not prove were in M, are those for which $f_c(z)$ does not have an attracting cycle, but does have an indifferent cycle which is either "parabolic" or a "Siegel ring."

More terminology [5]: We say that an analytic function f has an "indifferent cycle" iff there exists $z \in \mathbf{C}$ and a positive integer n such that $f^{[n]}(z) = z$ and $|f^{[n]'}(z)| = 1$. That cycle is "parabolic" if $f^{[n]'}(z) = e^{2\pi i p/q}$ for some rational p/q. We won't define Siegel ring, but it will suffice that C.L.Siegel in 1942 [30][27][5] showed that an analytic function f has a Siegel ring at a fixpoint z if $f^{[n]'}(z) = e^{2\pi i\alpha}$ where α is "not extremely well approximated by rationals," that is, for some fixed $\mu > 0$ and for all integers p, q with $q > 0$ sufficiently large, $|\alpha - p/q| > q^{-\mu}$.

Now, clearly, by a finite number of arithmetic operations with algebraic numbers we may prove that f_c has an indifferent cycle if it has one. If z is algebraic, then $f_c^{[n]'}(z)$ is too. It is a theorem ([1] chapter 3 "lower bounds for linear forms in logarithms" and supplementary material page 155) that if A is algebraic, then $(\ln A)/\pi$ is either rational, or isn't extremely well approximated by rationals. (This is, essentially, a quantitative version of a special case of the Gelfand-Schneider theorem after we note that $e^{i\pi} = -1$.) □

Theorem 15 (BCSS undecidability of Mandelbrot set) *The set of (x, y) in $\mathbf{R}^2$ such that $c = x + iy$ is in the Mandelbrot set, and c is on the boundary of the "main cardioid"*

$$(16x^2 + 16y^2 - 1)^2 < 4(1 - 8x + 16x^2 + 16y^2) \qquad (10)$$

is non-CUAS, and hence BCSS-undecidable.

Proof. Let "♡" denote the main cardioid EQ (10). We are going to prove that a dense full measure subset of ∂♡ lies in M, while another dense (but measure zero) subset of ∂♡ is outside M, from which the theorem immediately follows. ♡ is precisely the set of $c \in \mathbf{C}$ such that f_c has an attractive fixpoint [5]. We claim that its boundary ∂♡ consists entirely of c such that f_c has no finite attractive cycle, but f_c has a finite fixpoint which is "indifferent" – neither attractive nor repulsive. (The fact that f_c has no attractive cycle follows from a theorem of Shishikura [28] ([5] theorem 9.6.1 p.210) that there can be at most $2d - 2$ cycles in all when iterating a rational function of numerator and denominator degrees $\leq d$ on the Riemann sphere. Hence for f_c there can be at most 2 cycles in all. We know one of them is ∞, and one of them is the finite indifferent fixpoint, so there cannot be any others.)

Now among these c, if c is algebraic, we have previously shown (in the proof of theorem 14) $c \in M$. Indeed, the Siegel condition proves $c \in M$ not only for algebraic c, but in fact for a full measure set of c on ∂♡. On the other hand, a 1938 theorem of H.Cremer [16] ([27] p.30) shows that if a fixpoint of an analytic function f is z with $f(z) = z$ and $f'(z) = \lambda$, $|\lambda| = 1$, then if $\liminf_n |\lambda^n - 1|^{1/n} = 0$, then $c \notin M$. In our case $f(z) = f_c(z) = z^2 + c$ and $f_c'(z) = 2z$. Both these sets clearly are dense on the cardioid surface. The Siegel set obviously is dense since if x is algebraic and $(x, y) \in \partial\heartsuit$, then $x + iy$ is in the Siegel set. The Cremer set is dense because it contains any point of $\partial\heartsuit$ which is a limit point of a sequence of rationals with sufficiently rapidly increasing denominators, and $\heartsuit$ is the transformation of the circle $|c| < 1/2$ (which has a dense set of rational points on its surface, corresponding to the Pythagorean triples [21]) under the map $c \to c - c^2$.

Hence, the cardioid's surface, which is an algebraic curve, contains two dense sets: one (which is a superset of all the algebraic points on $\partial\heartsuit$) of points in M, another of points not in M. Therefore, each semi-algebraic set containing only points $c \in M$ must intersect $\partial\heartsuit$ only at a finite number of points, all of them algebraic, hence any countable union of such sets must leave all the non-algebraic Siegel points on $\partial\heartsuit$ uncovered. Hence $M \cap \partial\heartsuit$ is not CUAS. □

Theorem 16 (Exponential function) *The sets $y < e^x$ and $y > e^x$ are BCSS-decidable. The set $S_e = \{x, y | y \le e^x\}$ is BCSS-undecidable because it is non-CUAS. But if x and y are known to be algebraic numbers then the set $y \le e^x$ is decidable.*

Proof. To prove $y < e^x$ and $y > e^x$ are BCSS-decidable, we can use the continued fraction expansion ([22] 4.2.40) of e^x to generate an infinite sequence of over and underestimates of e^x (even and odd convergents of the continued fraction) which converge to e^x. To prove $y \le e^x$ is decidable if x and y are known to be algebraic: Use the Lindemann-Weierstrass theorem ([1] p.6, [8]) that if x is algebraic, then e^x is not (unless $x = 0$) hence equality $y = e^x$ is *impossible* except for the trivially testable case $x = 0$, $y = 1$.

To prove $y \le e^x$ is not CUAS, use the theorem that the set of e^x, with x algebraic, has unbounded "algebraic dimension," (also known as "transcendence degree"). This is because F.Lindemann in 1882 sketched and K.Weierstrass showed (Baker [1] p.6, [8]) that $\exp(\alpha_j)$ are algebraically independent if the α_i are algebraic numbers linearly independent over the rationals. Thus any union of semi-algebraic sets $P(x, y) \le 0$ lying entirely inside S_e, where the coefficients in the P's are taken from some algebraically closed field with *any* finite number of generating constants, cannot include all points $y = e^x$, x algebraic, on their boundaries. □

6 Open questions

1. Presumably for functions $g(z)$ of our form (EQ 1) it also is Turing-undecidable whether *all* z (except for a zero-measure set) get attracted to (or do not get attracted to) ∞, or some specified finite point; and presumably it is uncomputable *how many* attractors there are.

2. We have based our construction on a 2-counter machine, which emulates N steps by a universal Turing machine in $30^{2^{O(N)}}$ steps, and a 3-counter machine, which takes $2^{O(n)}$ steps.

Does there exist a meromorphic (or an entire) function whose iteration emulates N steps by a universal Turing machine (in a manner immune to noise of bounded magnitude) in only polynomial(N) *steps?* I have tried mightily to construct such a function, and failed; somehow the requirement for noise immunity always seems to conflict irreconcilably with the requirement for speed. So I suspect the answer is "no." A more precise variant of this question is: what is the minimum possible slowdown factor, as a function of N?

Although it is apparently unmentioned in the usual literature, it is easy to see that

Theorem 17 (Counter machines are slow) *A k-counter machine cannot emulate N steps of a universal Turing machine in $2^{o(Nk)}$ steps.*

Proof. The idea is that in n steps, a universal Turing machine can reach any of c^n different states, for some constant $c \ge 2$, whereas a k-counter machine can only reach $O(nk)$ states. To turn this idea into a proof, consider the task of "reading the input" where the input consists of N bits of information somehow pre-written on the Turing machine's tape (or tapes) and on the counter machines counters. A bit of input will be considered to be "read" only if altering that bit can alter the sequence of state transitions made by the machine. Clearly a Turing machine can read its input in $O(N)$ steps. The input to the counter machine must consist of at least one integer larger than $2^{N/k}$, and the most significant bit of that integer can only be detected after at least $2^{N/k}$ decrementing steps. □

It is readily seen that this bound is tight if $k \ge 3$. But I do not know if exponential time simulation is possible with only $k = 2$ counters; perhaps double-exponential slowdown, as in the proof of theorem 1, is best possible?

There may be some tradeoff between the speed of the g-iteration "computer" and the "wildness" of $g(z)$. For example, although our construction in theorem 4 yielded an entire function h which could emulate a universal Turing machine with only exponential, rather than doubly-exponential (as in theorem 2), slowdown; this was only accomplished at the cost of having a doubly exponential essential singularity at ∞, as opposed to the "tamer" essential singularity featured by the g in theorem 2. One measure [11] of the wildness of $g(z)$ as $|z| \to \infty$ is the "repetition counting function," saying the maximum number of times, as a function of R, that a value of $g(z)$ is repeated in $|z| < R$. This measure grows only polynomially with R for the $g(z)$ in theorem 2, but for the $h(z)$ in theorem 4, it grows exponentially.

3. Some open questions about Mandelbrot and Julia sets follow.

Does the boundary of M have measure 0? Is M locally connected? Are all Julia sets of rational functions nec-

essarily of measure 0 (if they are not the whole complex plane)? What is the asymptotic behavior of the area of the locus of points within distance ϵ of ∂M, as $\epsilon \to 0$?

Douady and Hubbard [17][5] proved M is simply connected. However this does not suffice to prove that M is locally connected, nor that ∂M has measure 0. For example:

Theorem 18 (Nasty boundary) *A bounded, open, simply connected set L (which is the union of a countable number of open algebraic sets) exists in the plane whose boundary has nonzero measure.*

Proof. Enumerate the rational points in the disc $|z| \le 100$: $q_1, q_2, \ldots$. Draw a line segment from 0 to each such rational point. The union of such line segments (the "star") is simply connected. For each real $s \ge 0$ we associate a unique point $z(s)$ on the star as follows: walk outward along the first line segment until you reach its end. Then walk outward along the second, then the third, and so on, until the total distance you have walked is s, then stop. Let L be the locus of points in $\mathbf{C}$ at distance $< \epsilon/(1+s/\epsilon)^2$ from $z(s)$. This locus is open, simply connected, contains all rational points, and has measure $< 12\epsilon^2$. Hence the complement of L within $|z| \le 100$ is closed and has empty interior (because it contains no rational points). Hence it is the boundary of L – but it has measure $> 100 - 12\epsilon^2$. □

Acknowledgement. This example was produced with assistance from Henry Cejtin.

A The lemniscate sine function sl

This section is a list of beautiful properties of $\mathrm{sl}(z)$, the inverse function of

$$\frac{1}{4K}\int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^4}} \tag{11}$$

where

$$K = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} \approx 1.311028778 \tag{12}$$

is the "lemniscate constant." Sl was the first elliptic function to be studied (by C.F.Gauss), but is rarely discussed nowadays. The integral in EQ (11) may be rewritten as

$$\sqrt{2}\int_0^y \frac{\mathrm{d}t}{\sqrt{1+t^4}},\ \ x^2 = \frac{2y^2}{1+y^4} \tag{13}$$

or as

$$\int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-2\sin^2\theta}},\ \ x = \tan\phi. \tag{14}$$

K may alternatively be expressed as

$$K^2 = \frac{\pi^2}{2}\sum_{n\ge 0}[\binom{2n}{n}4^{-n}]^3 \tag{15}$$

$$K = \sqrt{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t^4}} = \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-2\sin^2\theta}}, \tag{16}$$

$$(2K)^4 = \sum_{\substack{n,m\in\mathbf{Z}^2\\(n,m)\ne(0,0)}} \frac{15}{(n+mi)^4}, \tag{17}$$

$$K = \frac{\pi}{2\sqrt{2}}\left(1+\sum_{n\ge 1}[\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}]^2 2^{-n}\right). \tag{18}$$

For positive real a, b, let $\mathrm{AGM}(a,b)$ be the common limit of the sequences $a = a_0, a_1, a_2, \ldots$ and $b = b_0, b_1, b_2, \ldots$ defined when $n \ge 0$ by

$$a_{n+1} = \frac{a_n + b_n}{2},\ \ b_{n+1} = \sqrt{a_n b_n}. \tag{19}$$

Then

$$2K = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} = \frac{\pi}{AGM(1,\sqrt{2})}. \tag{20}$$

Sl has the following symmetries:

$$\mathrm{sl}(-z) = -\,\mathrm{sl}(z),\ \ i\,\mathrm{sl}(z) = \mathrm{sl}(iz),\ \ \mathrm{sl}(\bar{z}) = \overline{\mathrm{sl}(z)}, \tag{21}$$

$$\mathrm{sl}(z+\frac{1}{2}) = -\,\mathrm{sl}(z),\ \ \mathrm{sl}(z+1) = \mathrm{sl}(z+i) = \mathrm{sl}(z), \tag{22}$$

$$\mathrm{sl}(z+\frac{i+1}{4})\ \mathrm{sl}(z) = -i; \tag{23}$$

it obeys the addition formula

$$\mathrm{sl}(a+b) = \frac{\mathrm{sl}(a)\sqrt{1-\mathrm{sl}(b)^4} + \mathrm{sl}(b)\sqrt{1-\mathrm{sl}(a)^4}}{1+\mathrm{sl}(a)^2\,\mathrm{sl}(b)^2} \tag{24}$$

(which notice gives $\mathrm{sl}(2a)^2$ as a *rational* function of $\mathrm{sl}(a)^2$, and $\mathrm{sl}(2a)^4$ as a rational function of $\mathrm{sl}(a)^4$) and the half-angle formula

$$\mathrm{sl}(\frac{z}{2})^2 = \frac{1-\sqrt{1-\mathrm{sl}(z)^2}}{1+\sqrt{1+\mathrm{sl}(z)^2}}. \tag{25}$$

Its Maclaurin series expansion is

$$\mathrm{sl}(z) = y - \frac{y^5}{10} + \frac{y^9}{120} - \frac{11y^{13}}{15600} + \frac{211y^{17}}{3536000}\cdots \tag{26}$$

where $y = 4Kz$. The points

$$z = \frac{a+bi}{4}, \tag{27}$$

for integer a, b of the same parity, are precisely the set of zeros (when a, b are both even) and poles (when a, b are both odd) of $\mathrm{sl}(z)$.

The conformal map $w = \mathrm{sl}(z/4)$ bijectively maps the 45° rotated square with vertices $1, i, -1, -i$ to the unit disc $|w| < 1$, with each quadrant of the square being mapped to the same quadrant of the circle. The unit square with vertices $0, 1, i+1, i$ is bijectively mapped to the quarter-plane $\mathrm{Re}\,w, \mathrm{Im}\,w > 0$; the vertices are mapped to $0, 1, \infty, i$ respectively.

The conformal map $w = \mathrm{sl}(z/4)^2$ bijectively maps the unit square with vertices $0, 1, i+1, i$ to the halfplane $\mathrm{Im}\,w > 0$; the vertices are mapped to $0, 1, i\infty, -1$ respectively. The line segment with vertices $1, i$ is mapped to the semicircular arc $|w| = 1$; the line segment with vertices $0, 1+i$ is mapped to the positive imaginary axis.

Sl may be expressed in terms of the Jacobi and Weierstrass elliptic functions $\mathrm{sn}(z|k)$ and $P(z|g_2, g_3)$ respectively [22][25][33]. Specifically,

$$\mathrm{sl}(z) = \mathrm{sn}(4Kz|i) \tag{28}$$

arises from the special case of the Jacobian elliptic function sn with $k = i$, $k' = \sqrt{2}$, $K = K(i) = K(1/\sqrt{2})/\sqrt{2} \approx 1.311028778$, and $K' = K(\sqrt{2}) = (1-i) \cdot K$. Since $P(z|g_2, g_3)$ is the functional inverse of

$$\int_\infty^x \frac{\mathrm{d}u}{\sqrt{4u^3 - g_2 u - g_3}}, \tag{29}$$

and we may see (by considering the change of variables $t^{-2} = u$, $-2t^{-3}\mathrm{d}t = \mathrm{d}u$) that

$$\int_0^{1/\sqrt{x}} \frac{\mathrm{d}t}{\sqrt{1-t^4}} = -\int_\infty^x \frac{\mathrm{d}u}{\sqrt{4u^3 - 4u}}, \tag{30}$$

we see that

$$\mathrm{sl}(\pm iz)^{-2} = \mathrm{sl}(\pm z)^{-2} \tag{31}$$
$$= P(\pm 4Kz|4, 0) = 2P(\pm 4Kz\sqrt{2}|1, 0).$$

(Changing the four $\pm$ signs independently preserves validity.) Three well known representations of P and sn respectively then yield

$$(4K)^2 \,\mathrm{sl}(z)^{-2} = z^{-2} + \tag{32}$$
$$\sum_{\substack{n,m \in \mathbf{Z}^2 \\ (n,m) \neq (0,0)}} \left(\frac{1}{(z-m-ni)^2} - \frac{1}{(m+ni)^2} \right)$$

and

$$e^{\pi/4} \,\mathrm{sl}(z) = 2\sin(2\pi z) \cdot \tag{33}$$
$$\prod_{n \geq 1} \frac{1 - 2Q^{2n}\cos(4\pi z) + Q^{4n}}{1 + 2Q^{2n-1}\cos(4\pi z) + Q^{4n-2}}$$

$$= -2\frac{\sum_{n \geq 1} (-1)^n Q^{(n-1)n} \sin([4n-2]\pi z)}{1 + 2\sum_{n \geq 1} Q^{n^2} \cos(4n\pi z)} \tag{34}$$

where $Q = e^{-\pi}$.

Fast algorithms based on the "Landen transformation" for multiprecision evaluation of sn are described in [12] ex.7.3.2 p.227-228 and ex.1.4.5 p.17-18. See also [22][24][33].

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