On the enumeration of inscribable graphs

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Abstract — **We explore the question of counting, and estimating the number and the fraction of, inscribable graphs. In particular we will concern ourselves with the number of inscribable and circumscribable *maximal planar graphs* (synonym: simplicial polyhedra) on V vertices, or, dually, the number of circumscribable and inscribable *trivalent* (synonyms: 3-regular, simple) polyhedra. For small V we provide computer-generated tables. Asymptotically for large V we will prove bounds showing that these graphs are exponentially numerous, but, viewed as a fraction of all maximal planar graphs, they are exponentially rare.**

Many of our results are based on a lemma, the "strong 0-1 law for maximal planar graphs," of independent interest.

This is part of a series of TMs exploring graph-theoretic consequences of the recent Rivin-Smith characterization of "inscribable graphs." (A graph is a set of "vertices," some pairs of which are joined by "edges." A graph is "inscribable" if it is the 1-skeleton of a convex polyhedron inscribed in a sphere, "circumscribable" if it is the 1-skeleton of a convex polyhedron each face of which is tangent to a common sphere.)

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1 Introduction

A graph is "polyhedral" iff it is 3-connected and planar, or, equaivalently (the equivalence is a famous theorem of Steinitz [13]) iff it is the graph formed by the edges of a convex polyhedron. (A graph is "k-connected" if removing fewer than k vertices does not suffice to disconnect the graph. For the purposes of this paper, the triangle *does* count as a polyhedral, and as a 4-connected, graph.) We will use O, Ω, Θ asymptotic notation in this paper following Knuth.

It was recently shown by Rivin and Smith [15] that

Theorem 1 *(Rivin-Smith [15]) A polyhedral graph G is circumscribable if and only if weights w can be assigned to its edges such that*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each face f of G, the sum of the weights of all edges on the boundary of f is equal to 1.*

(W3) *For each non-facial cycle C in G, the total weight of all edges in C is strictly greater than 1.*

A graph is circumscribable iff its planar dual is inscribable.

(The "planar dual" $G*$ of a graph G has a vertex corresponding to each face of G, and vertices of $G*$ are joined by an edge if and only if the corresponding faces of G abutted.)

Using this Rivin-Smith characterization as our starting point, we intend to explore the questions:

1. How *many* V-vertex inscribable and circumscribable graphs are there?
2. What is the *probability* that a V-vertex simplicial polyhedron of random combinatorial type is inscribable or circumscribable?

To summarize our answers:

1. Exponentially many (see theorem 2).
2. Goes to zero exponentially (see theorem 6).

Along the way, we will prove a very useful result, the "strong 0-1 law for maximal planar graphs" (theorem 3), which often enables one to show instantly that the probability that a V-vertex maximal planar graph has some property, is exponentially close to 0 or 1, and indeed it yields bounds on the growth ratio of the exponential. A list of several such instant conclusions is presented as our corollary 4.

We conclude with table 2 of computer counts generated by inscribability testing algorithms, some miscellaneous remarks, and some open problems.

2 Asymptotic bounds for V large

Before saying anything about the asymptotic number of *inscribable* and *circumscribable* maximal planar graphs, it behooves us to recount what is known about the asymptotic number of maximal planar graphs ("MPGs") with V vertices.

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W.T. Tutte, in a famous paper [16], enumerated the *edge-rooted* planar triangulations (planar graphs with all faces triangular, except possibly for one "outer" face, whose order was fixed) on V vertices. Such graph is said to be edge-rooted if a particular vertex, edge, and face (all coincident) are colored red. A MPG without non-identity automorphisms and E edges thus has $4E$ possible edge-rootings. If it has an nontrivial automorphism group, however, then it has fewer (but nevertheless, at least one) distinct edge-rootings, since some of the $4E$ rootings are isomorphic.[1]

Some of Tutte's results follow. Let the number of edge-rooted MPGs with $m+3$ vertices be g_m and let the number of edge-rooted 4-connected MPGs with $m+3$ vertices be h_m. Define the generating functions

$$\begin{aligned} g(x) &= \textstyle\sum_{m\geq 0} g_m x^m = 1 + x + 3x^2 + 13x^3 + \\ &+68x^4 + 399x^5 + 2530x^6 + 16965x^7 + \\ &+118668x^8 + 857956x^9 + 6369883x^{10} + \ldots, \end{aligned} \tag{1}$$

$$\begin{aligned} h(y) &= \textstyle\sum_{m\geq 0} h_m y^m = 1 + y + y^3 + 3y^4 + 12y^5 + 52y^6 + \\ &+241y^7 + 1173y^8 + 5929y^9 + 30880y^{10} + \ldots. \end{aligned} \tag{2}$$

Then $g(x)$ and $h(y)$ are defined by the following equations. Let θ be the least real root of the quartic equation

$$x = (1-\theta)^3\theta; \quad \text{then } g(x) = \frac{(1-2\theta)\theta}{x} = \frac{1-2\theta}{(1-\theta)^3}. \tag{3}$$

Alternatively, let θ be the least real root of the cubic equation

$$(1-\theta)^3 y = (1-2\theta)^2\theta; \quad \text{then } h(y) = 2 - \frac{(1-\theta)^3}{1-2\theta}. \tag{4}$$

Rather ugly explicit formulae for $g(x)$ and $h(y)$ could be produced with the aid of the quartic and cubic formulae. Tutte found the remarkably simple formula

$$g_m = \frac{2(4m+1)!}{(m+1)!(3m+2)!}. \tag{5}$$

A corresponding (but much messier) explicit expression for h_m may be produced with the aid of the Lagrange-Bürmann series inversion formula.[2] In practice it is eminently satisfactory to use a Newton iteration on formal power series, starting with the first few terms as an initial guess, to "solve" said quartic or cubic. Observe that Tutte regards the 3-vertex "doubled triangle" as a 4-connected MPG, by convention.

It then follows from (EQ 5) by using Stirling's formula that

$$g_n \sim \frac{1}{16}\sqrt{\frac{3}{2\pi}}n^{-5/2}(\frac{256}{27})^{n+1}, \tag{6}$$

and a slightly harder asymptotic analysis gives

$$h_n \sim \frac{1}{128}\sqrt{\frac{3}{\pi}}n^{-5/2}(\frac{27}{4})^{n+1}. \tag{7}$$

In fact, the exponential growth ratios 256/27 and 27/4 *alone* may be deduced quite simply from a singularity analysis of $g(z)$ and $h(z)$ in the complex plane. One observes that when $x \to 27/256-$, the two least real roots of the quartic collide at $\theta = 1/4$ and then become a complex conjugate pair; similarly as $y \to 4/27$ the two least real roots of the cubic hit, also at $\theta = 1/4$, becoming a complex conjugate pair. (Nothing non-smooth happens within $|y| < 4/27$, however.) More generally, the points of nonanalyticity of the roots of a polynomial $P(t)$ (whose coefficients are analytic functions of a parameter x) as a function $t(x)$, can only occur where $D(x) = 0$ where $D(x)$ is the discriminant [19] of $P(t)$. In the present cases: the discriminant of the quartic is $(27-256x)x^2$ and the discriminant of the cubic is $-y(27y-4)(y+4)$.

Now Tutte [17] and others [14] [1] later showed that only an exponentially small fraction [$< .9971^V$ (!)] of V-vertex MPGs have a nontrivial automorphism group. This implies that the correct asymptotic formula for the number of MPGs may be obtained by dividing the above expression for g_n by $4E$, where $E = 3n+3$. One would similarly conjecture (although I am unaware of a proof) that the correct asymptotic formula for the number of 4-connected MPGs would be obtained by dividing the above expression for h_n by $4E$. *Whether or not this conjecture is true* [and nobody doubts it!], the correct expressions are obtained by dividing by some quantity between 1 and $4E$ inclusive, so that, thanks to Tutte, we *do* know the correct orders of the exponentials, and dividing by $4E$ does yield a lower bound.

Since every *4-connected* MPG is both inscribable and circumscribable (by a theorem of Dillencourt and Smith [5] [7]) we conclude

Theorem 2 [Asymptotic lower bounds]. *The number of V-vertex MPGs is asymptotic to*

$$\frac{V^{-7/2}}{64\sqrt{6\pi}}(\frac{256}{27})^{V-2}, \tag{8}$$

and the number of them that are inscribable, and the number of them that are circumscribable, are both greater than a quantity – corresponding to the number that are 4-connected – asymptotic to

$$\frac{V^{-7/2}}{512\sqrt{3\pi}}(\frac{27}{4})^{V-2}. \tag{9}$$

□

[1] Tutte introduced edge-rooting to avoid having to worry about automorphisms when counting graphs. Nevertheless, it ought to be possible to enumerate MPGs according to their symmetry group, using methods similar to Tutte's, and then to combine these results to achieve an exact enumeration of the unrooted graphs. This project might require approximately 20 times as much work as Tutte did. For some results in these directions, see [2].

[2] Apparently the true reason why such a simple formula exists for g_m (but does not exist for h_m) is because of the existence of the oddball expansion

$$\theta = \frac{x}{(1-\frac{x}{(1-\frac{x}{(1-\frac{x}{(1-\ldots)^3})^3})^3})^3}.$$

One may observe (from the algebraic expressions above) that $g(x)$ and $h(y)$ obey the relation

$$g(x) = \frac{1}{2 - h(xg(x)^2)}. \tag{10}$$

This identity may also be shown combinatorially by observing that any edge-rooted MPG with $m+3$ vertices ($m > 0$ of them "internal") may be obtained by "pasting" rooted MPGs with m_i internal vertices inside each non-root face of a 4-connected rooted MPG with $n > 0$ internal vertices (and $2n+1$ internal faces), where $n + \sum_{0 \le i \le 2n} m_i = m$. This leads to $g(x) = 1 + \sum_{n \ge 1} h_n g(x)^{2n+1} x^n$, which implies $2g(x) = 1 + g(x)h(xg(x)^2)$, which gives (EQ 10).

We are now going to use a similar identity to obtain a powerful general theorem, which we will call the "strong 0-1 law for MPGs."

If G and H are edge-rooted maximal planar graphs, we say that H is a *strictly embedded subgraph* of G if, in a planar drawing of G whose root face is the external face, the part of the drawing lying inside and on a separating triangle T of G is a drawing of H whose root face is T.

Theorem 3 [Strong 0-1 law for maximal planar graphs] *Let there be a set of $R > 0$ distinct edge-rootings $Q_1, Q_2, \ldots, Q_R$ of particular fixed maximal planar graphs, where Q_i has $K_i + 3$ vertices, $K_i > 0$. Let $p(x) = \sum_{1 \le i \le R} x^{K_i}$ be their generating polynomial. Then among V-vertex rooted maximal planar graphs, all but an exponentially small fraction contain some Q_i as a strictly embedded subgraph. In fact, the number q_{V-3} of V-vertex edge-rooted MPGs which do not contain a Q_i as a strictly embedded subgraph is $\Theta(V^{-5/2}c^V)$, where c, $1 < c < 256/27$, is the greatest real root of*

$$[1 - p(\frac{1}{c})]^2 = \frac{27c}{256}. \tag{11}$$

If $p(x) = Rx^K$ with R fixed and K large, then $\frac{256}{27} - c \sim (\frac{27}{256})^{K+1} \cdot 2R$.

A short table follows in which the values of c have been rounded to 5 decimals and $p(x) = Rx^K$.

R	K	c
0	any	256/27 = 9.481481...
1	1	6.94893
1	2	9.26170
2	2	9.02118
3	2	8.75359
1	3	9.45909
2	3	9.43640
3	3	9.41340
1	4	9.47913
2	4	9.47678
3	4	9.47442
1	5	9.48123
2	5	9.48099
3	5	9.48074
4	5	9.48049
1	6	9.48146

Table 1: Values of c in theorem 3.

Proof. Let $q(x)$ be the generating function $\sum_{m \ge 0} q_m x^m$ counting edge-rooted MPGs with m internal vertices which do not contain a strictly embedded subgraph Q_i. Then

$$q(x) = 1 + \sum_{n \ge 1} h_n [q(x) - p(x)]^{2n+1} x^n \tag{12}$$

so that

$$2q(x) - p(x) = 1 + [q(x) - p(x)] h([q(x) - p(x)]^2 x). \tag{13}$$

Upon letting θ denote the least real solution of the cubic

$$(1-\theta)^3 [q(x) - p(x)]^2 x = (1 - 2\theta)^2 \theta, \tag{14}$$

so that

$$q(x) - p(x) = (1 - 2\theta) \sqrt{\frac{\theta}{(1-\theta)^3 x}} \tag{15}$$

we have

$$h([q(x) - p(x)]^2 x) = 2 - \frac{(1-\theta)^3}{1 - 2\theta} = \frac{1 - \theta - 3\theta^2 + \theta^3}{1 - 2\theta} \tag{16}$$

so that

$$2(1-2\theta)\sqrt{\frac{\theta}{(1-\theta)^3 x}} - (1 - \theta - 3\theta^2 + \theta^3)\sqrt{\frac{\theta}{(1-\theta)^3 x}} = 1 - p(x) \tag{17}$$

Squaring both sides and multiplying by x yields

$$(1-\theta)^3 \theta = [1 - p(x)]^2 x \tag{18}$$

Upon solving this quartic for θ in terms of x we may then express $q(x)$ in terms of x. Without going through this procedure we can still do a singularity analysis of $q(x)$ by computing the discriminant

$$D(x) = (27 - 256Z)Z^2, \quad Z = [1 - p(x)]^2 x \tag{19}$$

of this quartic. The appropriate root r of $D(x)$ is the unique odd-multiplicity (multiplicity 1, in fact), positive, real root of

$$[1 - p(x)]^2 x = 27/256 \tag{20}$$

which corresponds again to the value $\theta = 1/4$. Observe that $x > 27/256$. (Any roots corresponding to $p(x) = 1$ are double. The unique single root has $x > 27/256$ because clearly $x > 0$, and it cannot be that $0 \le x \le 27/256$ because $0 \le p(x) < g(x) \le g(27/256) = 32/27 < 2$ in this range.) The value of c in the theorem is $c = 1/x$.

The expansion of $\theta - 1/4$ near r may be seen to behave proportionally to $(x - r)^{1/2}$, since it arises from the collision of two real roots of the quartic, which then become a complex conjugate pair, analagously to the behavior of the roots t of $t^2 - xt - 1$ as x increases through 2. (Observe that no point of nonanalyticity of $\theta(x)$ arises from a *double* root of $D(x)$, analagously to the behavior of the roots t of $t^2 - (2 - x^2)t - 1$ as x increases through 0. This is why we had excluded double roots from consideration. Of course a triple, quintuple, etc. root could conceivably occur but this would still be the unique odd-multiplicity root and our results would still be valid in this case.)

Upon expanding $q(x)$ about this point in powers of $\theta - 1/4$ one observes that there is *no* linear term, but there is a cubic term, so the nonanalyticity of $q(x)$ arises from a term $(x-r)^{3/2}$. Upon now applying the "transfer lemma" of Flajolet and Odlyzko [11] we obtain the $V^{-5/2}$ term in the result.

(Incidentally it is possible to write a formula for q_m by following a procedure similar to Tutte's Lagrange inversion determination of g_m. We omit details.)

Finally, we observe that the length $S(x)/S'(x)$ of a Newton step, starting from $x = 27/256$, to find the root of the polynomial $S(x) = (1 - Rx^K)^2 x - 27/256$, is

$$\frac{27R}{256} \cdot \frac{2[27/256]^K - R[27/256]^{2K}}{(1 - R[27/256]^K)(1 - R[2K+1][27/256]^K)}. \quad (21)$$

Because if R is fixed and K is large the value of $S'(x)$ within the interval between $27/256$ and the root is $1 + O((K+1)R(27/256)^K)$, we see that the Newton step length is asymptotic to the true length of this interval. This leads to the asymptotic expression for c. □

It follows almost instantly from theorem 2 that

Corollary 4 *Among the V-vertex edge-rooted maximal planar graphs, the percentage having any of the properties on the list below, is $O(c^V)$ for some $c < 1$.*

1. *Hamiltonian.*
2. *Dual is Hamiltonian.*
3. *3-colorable.*
4. *Has a perfect matching.*
5. *Has a k-factor (for some fixed $k > 0$).*
6. *1-tough (see definition 7).*
7. *Has no vertex of valence k (for some fixed $k > 2$).*
8. *Inscribable.*
9. *Circumscribable.* □

Indeed, suitable values of c are readily computed. We mention that the *true* growth ratio for counting MPGs with no 3-valent vertices, (or, dually, triangle-free simple polyhedra) is ≈ 6.948932. This result appears new.

To focus in on cases 8 and 9:

Observation 5 [**Bad graphs**]

1. *A maximal planar graph containing an octahedron as a strictly embedded subgraph, is not circumscribable.*
2. *A maximal planar graph containing the 7-vertex graph obtained by stellating the 3 "interior" faces of a tetrahedron, as a strictly embedded subgraph, is not inscribable.*

Proof. This follows immediately from theorem 1:

1. We claim no polyhedral graph obtained by drawing stuff inside exactly one face "f" of an octahedron, can be circumscribable. If it were circumscribable, then the edges of the boundary of f would have total weight > 1, forcing the antipodal octahedron face $\bar{f}$ to have (illegal) total weight < 1, due to the 6 equalities forcing the total weight of each of the 6 other octahedron faces to be 1.
2. We claim that any polyhedral planar graph obtained by drawing stuff inside all 4 of the faces of a tetrahedon T, cannot be inscribable. The total weight of the edges connecting the 3 vertices of a face of T to the stuff inside that face, must be ≥ 1. Hence, in order that the sum of the edge weights at the 4 vertices of T be 4, the average weight of an edge of T must be ≤ 0, which is illegal. □

In fact, more generally: **Extension.**

1. Any polyhedral graph constructed by starting with a k-gonal antiprism with $k \geq 3$, and adjoining some graph drawn entirely inside one of the two k-gons, is not circumscribable.
2. Any graph containing an un-inscribable MPG as a subgraph (strictly embedded or not) is un-inscribable. □

Using plain observation 5 (without the extension), plus theorem 2, and $(R, K) = (1, 3)$ and $(1, 4)$ respectively, we obtain

Theorem 6 [**Asymptotic upper bounds**] *The number of edge-rootings of circumscribable and inscribable maximal planar graphs with V vertices are $O(V^{-5/2}c^V)$, where $c \approx 9.45909$ and 9.47913 respectively. (More exact values of c may be obtained by solving the relevant polynomial equation from the strong 0-1 theorem.)* □

These values are less than the growth ratio $9.481481\ldots$ for the total number of MPG's (cf. theorem 2) but not by much! The author made an attempt to take the extension of observation 5 into account (using item 2 recursively) and after a lot of work, he was only able to obtain further improvements of order 0.0001.[3]

3 Computer counts for small V

Using methods of [4] and several independently written graph isomorphism testers, the present author and his computer have generated catalogues of all nonisomorphic

[3]It does not seem worth it to explain these improvements in any detail, since they are quite weak and since the arguments required are more complicated than the simple methods used here. We merely wish to point out that they exist. The fact that further improvement seems difficult perhaps indicates that we are near the truth; I dogmatically conjecture that the truth is greater than 9.

planar triangulations with ≤ 14 vertices. Michael Dillencourt [6], using a more complicated program, but with a less voracious appetite for disk space, independently did the same thing up to 16 vertices; these provided useful checks on each other's results.

These catalogues were used as input to various inscribability and circumscribability tests (which we will describe below) and the result was table 2.

V	#V-vertex MPGs	non-circmscbbl	non-inscribbl
4	1	0	0
5	1	0	0
6	2	0	0
7	5	1	0
8	14	2	1
9	50	8	1
10	233	35	9
11	1249	168	48
12	7595	999	[343, 383]
13	49566	6340	[2466, 2809]
14	339722	43133	at least 18905
15	2406841	305271	
16	17490241	2231377	

Table 2: Computer-generated graph counts. [] notation gives lower and upper bounds in cases where the exact number is not known.

It is amusing to observe that the fact that circumscribable and inscribable MPGs are exponentially rare, might never have been guessed from these computer results, which seem to be indicating the opposite! The first few examples of noninscribable and noncircumscribable graphs (which account for first few nonzero entries in the last two columns of table 2) all arise from observation 5.

We will now discuss the algorithms used to generate this table in greater detail.

It was shown by Rivin and Smith [15] that theorem 1 in fact leads to a polynomial time algorithm to decide whether a graph is inscribable or circumscribable. This algorithm was obtained by observing that theorem 1 is a *linear program*, and then after constructing certain mildly clever "separation oracles" by using subroutines for finding shortest cycles in graphs, and proving a few inequalities, we may apply the *ellipsoid algorithm* [12] (or P.M. Vaidya's more recent convex programming algorithm [18]) to solve it in polynomial time. Unfortunately, this algorithm of Rivin and Smith, while polynomial time, seems to require the use of high precision arithmetic (numbers with order $V \log V$ digits arise) and the best known upper bound on the running time is rather large – $O(V^{4.38})$ steps, each step involving a high-precision arithmetic operation. In fact, it turns out that the ellipsoid method for linear programming suffers from severe numerical instability, and high precision arithmetic is almost always *required* when using it. This fact is not too well known in the theoretical computer science community but was pointed out in [3].

Rivin and Smith were unfortunately unaware of this when they wrote the code – they were using fixed double precision arithmetic, and as a consequence their MPG tester became unreliable when $V \geq 10$.[4] Nevertheless it was possible by simple a posteriori tests, even with fixed precision arithmetic and subject to roundoff, to gauran-tee that the Rivin-Smith code only made *one-sided* errors, i.e. if it claimed it had found an inscription, it actually had; the only possible errors lay in making false claims of noninscribability. These claims were sufficiently rare that it was possible to test the few graphs in question, with $V \leq 10$, by hand.

Definition 7 *A graph is* 1-supertough *if the removal of some s vertices, $1 < s$, splits the graph into $< s$ connected components. (*1-tough: *replace the $<$'s with $\leq$'s.)*

It is known that all nonbipartite inscribable graphs are 1-supertough [7] [9] [15]; by examining the MPGs with $V \leq 10$ vertices we found that

Fact 8 *If $V \leq 11$, then a V-vertex maximal planar graph is inscribable if and only if it is 1-supertough.*

The additional verification of this statement for $V = 11$ was performed as follows: we found that of the 1249 11-vertex MPGs, exactly 48 were not 1-supertough, and of the remaining 1201 MPGs, all but 5 were 1-hamiltonian.

Definition 9 *A graph is* k-hamiltonian *if the removal of $k \geq 1$ vertices, always leaves a hamiltonian graph.*

It is known that all 1-hamiltonian planar graphs are inscribable [5]. We then examined the 5 graphs in question by hand and found weightings showing each was inscribable. We also found, as an independent verification of our inscribability testing of the 233 10-vertex MPGs, that all but 9 were 1-supertough, and all of the 224 1-supertough 10-vertex MPGs were 1-hamiltonian, with the *single exception* of the inscribable graph in figure 1.

We know that there are graphs, however, which are 1-supertough but non-inscribable. A 15-vertex example is shown in figure 2.

We conjecture that this is the unique minimal counterexample, so that, conjecturally, fact 8 would still remain valid – but for this single exception – if the "11" were replaced by "15."

The third column of Table 2 was obtained by Dillencourt [6], using an algorithm for recognizing inscribable

[4] Note to future implementors. We now believe the most practical way to implement the test in theorem 1 is to solve the linear program using a simplex method and only including constraints of forms W1,W2, and maximizing the "buffer" amount by which the constraints are satisfied. If no solution exists, the graph is noninscribable. If the solution found obeys W3, the graph is inscribable. If it disobeys W3, find a violated constraint and re-solve the linear program, augmented by this constraint. In practice, few augmentations will be required before a definite answer is reported. Admittedly, one still can find oneself in doubt if the best buffer value found is a very small number (is it $<, >, = 0$?) so that exact rational arithmetic can still become required, but usually only in the final stages of the simplex algorithm.

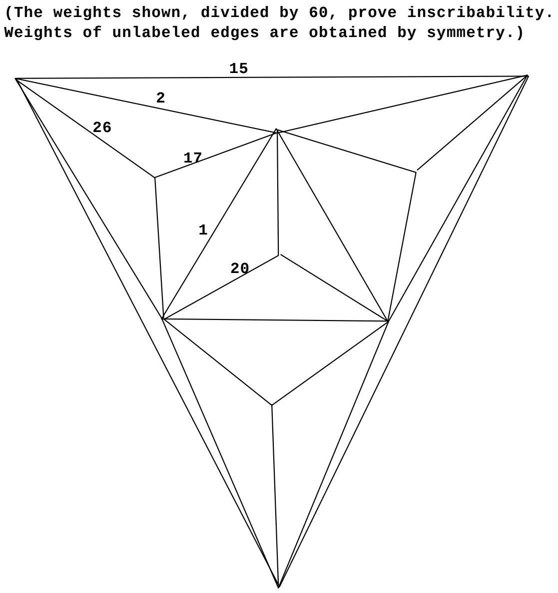

Figure 1: The minimal 1-supertough, non-1-hamiltonian maximal planar graph. Inscribability proof included in figure.

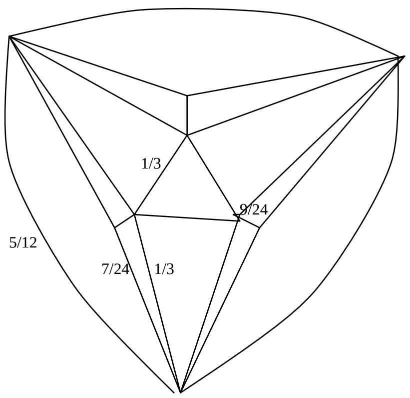

Figure 2: A non-inscribable, 1-supertough graph from [6]. Only half the graph is shown; the other half is identical and "lies on the other side of the page." The weights given prove this graph is *circumscribable.*

trivalent polyhedra developed by Dillencourt and Smith [?, 9]. This algorithm runs in linear time and requires only low-precision integer arithmetic. It is based on the following theorem [?, 9]:

Theorem 10 *A trivalent polyhedral graph is inscribable if and only if it is either 1-supertough, or bipartite and with 4-connected planar dual.* □

The lower bounds for $V = 12$-14 in the last column of table 2 arose from counting non-1-supertough MPGs; conjecturally, they are exact. The upper bounds arose from counting 1-hamiltonian MPGs.

4 Miscellaneous

It is also possible to construct exponential numbers of interesting kinds of inscribable graphs using any combination of the following four methods.

1. *4-connected polyhedra are inscribable. (Proven by Dillencourt and Smith [10], [5].)*
2. *Add diagonals to the faces of inscribable nonbipartite graphs (Dillencourt, Rivin, and Smith showed, in their "add-an-edge" theorem, that this operation always preserves inscribability [7], [5].)*
3. *Glue together different kinds of wire-frame hyperbolic ideal polyhedra, whose dihedral angles are acute, (equivalently: their edge-weights in theorem 1 are $> 1/4$) at congruent faces, in such a way that no three polyhedra share an edge. The resulting graph is inscribable. (As a consequence of the "hyperbolic connection" in Rivin and Smith [15].)*
4. *The "medial graph" M of a polyhedron P has vertices corresponding to the midpoints of the edges of P; two vertices of M are connected by an edge iff the two corresponding edges of P were adjacent on the same face. All medial graphs of all polyhedra are inscribable (this may be shown by using all edge-weights equal to $1/4$ in theorem 1).*

The third method, for example, shows that graphs gotten by building treelike structures by gluing congruent regular wireframe dodecahedra at common pentagons, or N-gonal wireframe prisms at common square faces (including with "twists"), or stacking any number of N-gonal wireframe prisms or antiprisms, are all inscribable.

As consequences of an NP-completeness proof of Dillencourt's [8], a nonHamiltonian inscribable (inscribable since its dual is 4-connected) 3-valent graph due to Walther [20] (also given in [13]), Whitney's theorem (stating that all 4-connected MPGs are hamiltonian) [21], and an easy construction (respectively), it may be shown that:

Theorem 11 **[Hamiltonian statements]** *There are exponentially many nonHamiltonian simplicial and simple inscribable polyhedra, and also exponentially many Hamiltonian simplicial and simple inscribable polyhedra.* □

5 Open problem

Find the exact growth rates of the exponentials for counting inscribable or circumscribable maximal planar graphs.

Considering the Dillencourt-Smith [9] simple, linear-time, graph-theoretic characterization of inscribable trivalent polyhedra, this may be within reach.

6 Acknowledgements

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