The Energy-Time Uncertainty Principle

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Abstract — **This paper contains a simple but powerful statement and proof of the Energy-Time uncertainty principle.**

Keywords — Energy-Time uncertainty principle. Maximum rate of time evolution. Schrödinger equation. Quantum mechanics.

1 Introduction and motivation

My motivation to write this paper came from reading [20]. In this and other papers, Landauer argues that the energy time uncertainty principle does *not* force energy dissipation to occur during communication. Although Landauer's papers are interesting and important, and may be correct, they suffer from the major problem that Landauer nowhere *defines* exactly what he thinks the energy-time uncertainty principle is, nor does he provide a simple precise mathematical definition of "communication." It seems quite plausible to me that with varying, but still reasonable, definitions of these notions, one could reach varying conclusions on this fundamental question. Another motivation is the question of fundamental limits on computer speed: our (EQ 10) and its ilk[1] might be interpreted as saying that logical "states" of a computer can't change too quickly. For example, if "qubits" means "2-state systems," (EQ 12) tells us that a 90° "rotate" operation on one qubit within some multibit physical system requires time $\geq \pi\hbar/(\overline{E-E_0})$. where the expected energy of the physical system is $\overline{E-E_0}$ above the ground state.

After reading [20] I asked myself: just what *is* the energy time uncertainty principle, anyhow? After looking in about 20 quantum mechanics books, I was astonished to find (this is 1998!) that only one of them [28] managed to give an acceptable statement of it, where by "acceptable" I mean an inequality in which all variables have a mathematical definition. I then managed to produce such a statement myself, and found to my surprise that it seems to be substantially simpler and more powerful than previous ones.

Section 2 gives a history of uncertainty principles including an amusing debunking of common myths about the Bohr-Einstein box debate. For another history quite complementary to ours, see [11]. Readers who just want the result may skip the historical section and go directly to section 3.

[1]Eq. 4 of [24] gives essentially the same result as our (EQ 12).

2 History

Most quantum mechanics books state and derive the $\Delta p_j \Delta x_k \geq \hbar\delta_{jk}/2$ uncertainty principle, where Δp_j means the square root of the variance of the jth momentum coordinate (as a random variable specified by a wavefunction) and Δx_k means the square root of the variance of the position x_k (ditto). Some go further and give a general uncertainty relation [27] among any two quantities corresponding to noncommuting operators[2]. Note: these relations are inequalities in which every term in the formula has a precise mathematical definition.

Almost all books mention the energy-time uncertainty principle, but apparently only Thirring's [28] actually gives an inequality in which every term in the formula has a mathematical definition. (He also gives a brief original proof.)

Theorem 1 (Thirring [28]) *Suppose Ψ is a wavefunction evolving with time t to $\exp(-i\hat{H}t/\hbar)\Psi$, where $\hat{H}$ is a Hamiltonian operator. Suppose the probability that an energy measurement on Ψ would yield a value in an energy interval of width ΔE, is $\geq 1-\delta$. Then*

$$\int_{-\infty}^{\infty} |\langle\Psi|e^{-iHt/\hbar}\Psi\rangle|^2 \mathrm{d}t \geq \frac{(1-\delta)^2}{h\,\Delta E}. \tag{1}$$

Unfortunately, the integral in Thirring's theorem often does not converge[3]. For example, consider the simplest possible case, where H is a constant. Then $\int_{-\infty}^{\infty} \mathrm{d}t = \infty$. We also get ∞ (since $\int_{-\infty}^{\infty} |\mathrm{TP}(t)|\mathrm{d}t = \infty$ if TP is any nonzero trigonometric polynomial) if Ψ is any linear combination of any N eigenfunctions with different eigenvalues of any Hamiltonian H. An upper bound which so often is infinite cannot be regarded as very useful; in contrast, all quantities in our (EQ 7) are finite.

The energy-time principle is of a different character from the usual ones, because time does not correspond to a Hermitian operator with a complete set of orthonormal eigenstates. Energy (e.g. for systems with Coulombic, or more generally $L^2 + L^\infty$, interparticle potentials) does [15] [17].

[2]Extensions of this include: other moments than 2 [10], and rather unclear claims about Shannon information [12]. Lieb [21] [22] has his own forms of the $\Delta p \Delta x$ uncertainty principle, based on Sobolev inequalities, which seem more useful for many purposes. For example (as Lieb shows) his inequality yields a rigorous bound on the ground state energy of the hydrogen atom, whereas the usual formulation yields nothing.

[3]A theorem very related to Thirring's (e.g. with the same left hand side) was given in [11].

Since time is just a real number, it commutes with everything.

There have been a few attempts in the literature[4] to formulate and prove energy-time uncertainty relations. Of these, the only ultrasimple proof besides mine and Thirring's [28] (at least, in the corrected form below) is Mandelstam and Tamm's [23]: From [27] we have

$$\Delta R\, \Delta S \;\ge\; |\overline{RS - SR}|/2 \tag{2}$$

where ΔR means the square root of the variance of R, and overline denotes expectation value, for any two operator observables R, S. The Ehrenfest expectation value version of Von Neumann's version of the Schrödinger time evolution equation is

$$\hbar\, \dot{\overline{R}} \;=\; -i\,(\overline{HR - RH}) \tag{3}$$

where H is the Hamiltonian. Hence

$$\Delta R\, \Delta H \;\ge\; \frac{\hbar}{2} |\dot{\overline{R}}|, \tag{4}$$

so that after time evolution for a time lapse t,

$$t(\Delta H)_{\max} \;\ge\; \frac{(\overline{R_{\text{new}} - R_{\text{old}}})\,\hbar}{2\,(\Delta R)_{\max}} \tag{5}$$

where the subscript "max" indicates the maximum value of the indicated standard deviation occurring throughout the time evolution of duration t. Hence

Theorem 2 (Corrected version of Mandelstam and Tamm [23]) *If we let $(\Delta t)_R$ denote the minimum amount of time required for the expectation value of R to change additively by $\pm(\Delta R)_{\max}$, i.e. by 1 of its own standard deviations, then*

$$(\Delta t)_R\, (\Delta H)_{\max} \;\ge\; \frac{\hbar}{2}. \tag{6}$$

The Mandelstam-Tamm theorem has the advantage that it works for *any* R, but it seems hard to use because $(\Delta R)_{\max}$ and $(\Delta H)_{\max}$ are rather strange quantities. The inequality (EQ 7) we are going to give is easy to use because it involves notions of ΔE arising from the wavefunction Ψ now. In contrast, the Mandelstam-Tamm result can be hard to use because $(\Delta R)_{\max}$ and $(\Delta H)_{\max}$ arise from an unknown wavefunction at an unknown later time.

Among the most amusing historical developments in the early history of quantum mechanics were the oft-recounted debates between A.Einstein and N.Bohr[5]. The legends associated with these debates are positively permeated with the same aura as the famous story[6] of George Washington and the Cherry Tree.

Einstein at the 1930 Solvay conference [4] [5] presented to Bohr the following attack on quantum mechanics via the energy time uncertainty principle. Keep in mind that in 1930, it was still 15 years before the first precise statement of this principle had been made [23], so Bohr and Einstein were arguing about an issue neither understood. (They both thought the uncertainty principle was $\Delta E \Delta t \ge \hbar/2$ and with the interpretations of the Δ's the same as in the $\Delta p \Delta x$ principle. This interpretation has never been justified, and also note that the coefficient $1/2$ on the right hand side is also unjustified, at least in theorem 3 and in [11].) Einstein considered a box being weighed by a spring scale and containing photons. A cuckoo-clock mechanism mounted on the box opens a door on the box's side for a time duration t, specified extremely precisely, and with a precision apparently independent of any quantity having to do with photon energies. The weight of the box before and after the measurement seems to be measurable extremely precisely, and hence, since $E = mc^2$, the energy of the escaping photons is deducible extremely precisely. Hence we get $\Delta E \Delta t$ (where ΔE and Δt denote uncertainties in the values of t and of the energy E of the photons that escaped through the door) arbitrarily small, "refuting" quantum mechanics. Of course, of the three formulations of the energy-time principle given here, and Finkel's [10], none of them pertain to "uncertainty in the time lapse" at all, so this entire idea of Einstein's was based on a false premise. But neither Einstein nor Bohr were aware of that, and so both thought this was a tremendously dangerous attack on the foundations of quantum mechanics. So then (the usual tale proceeds) Bohr after great effort refuted Einstein's attack by invoking general relativity (!), as follows. The uncertainty Δz in the vertical position z of the spring scale obeys $\Delta z \Delta p_z \;\ge\; \hbar/2$. Einstein presumably wants $\Delta p_z \ll g m_{\text{escaped photons}} t$, where g is the acceleration of gravity. But now, the clock mounted on the box, according to general relativity, differs by Δt from the time on a clock attached rigidly to the Earth, where $\Delta t \approx g t \Delta z / c^2$. Hence our knowledge of the time lapse will suffer from an un-

[4] [18] gives a $\Delta E \Delta T$ uncertainty principle, where T is *not* time, but rather a new operator called "tempus." Finkel [10] can obtain a true $\Delta E \Delta t$ principle concerning "RMS state lifetime" but needs to introduce probabilities of state transitions and results about them from a different paper, while our result does not need to assume any transitions (although it certainly still applies if there are any). Non-holonomy in Lie algebras is used by [14] in an extremely non-elementary paper without a clear statement of a physically meaningful result. Wigner's paper [30] seems to give also an exceedingly vague statement about "the spread in time of presence at a particular plane," whatever that is. Papers analyzing specific physical scenarios and which therefore cannot claim any universality include [7] and [25], although at least the latter is clearly written.

[5] For example, it's often recounted how Einstein, Podolsky, and Rosen mounted an attack [9] on quantum mechanics, which fortunately Bohr refuted [6]. This tale is nonsense, because Bohr's "refutation" is completely muddled, and if you or I had submitted it, would have been ripe for referee ridicule and rapid rejection. E, P, and R's reasoning is in comparison a masterpiece of clarity, and in fact the idea of "local hidden variable theories" their paper spawned (after additional clarification of what that might mean by later authors) was quite logically reasonable. Only much later was the full picture revealed. With the work of Kochen and Specker [19] it was seen that it was logically impossible for hidden variable theories always to give the same predictions as quantum mechanics. Then work on Bell's inequality [8] developed to the point where an experimental test could be, and was, carried out [2] [3], with the result supporting quantum mechanics and refuting local hidden variable theories.

[6] Entirely fictional: this fabrication created by Parson M.W.Weems [29] in 1800 has forever after been inflicted upon US school children in an effort to make them realize the importance of honesty.

certainty of order $h/(m_{\text{escaped photons}}c^2)$, saving the uncertainty principle $\Delta E \Delta t \gtrsim h$ and quantum mechanics. The fairy tale[7] concludes: Einstein was so stunned at this use of his own theory of general relativity against him, that he conceded defeat.

Actually, Bohr's counterattack, although startling, had two fatal flaws. First, I do not see what Δp_z has to do with $g m_{\text{escaped photons}} t$. We could agree only to perform weighings after exponentially damping the box's motion by placing it in a bath of viscous oil; any uncertainties in the weighing would then seem to be of a fixed magnitude exponentially independent of anything else. (The box would be in a minimum uncertainty state after the damping, roughly.) Second, neither gravitational time dilation nor Δz need have anything to do with the case. This is because Einstein could instead have measured the mass of the box with an electric field (after charging the box with a known amount of charge). The capacitor plates generating the electric field could have been superconductively shorted during the door-cycling (so that the box would be exposed to precisely zero field during the door-cycling, avoiding all possibility of time dilation caused by Δz during that epoch) but charged to 1 volt for the weighings – say by connection to an arbitrarily enormous external capacitor bank providing a 1 volt reference. Thus, both steps in Bohr's argumentation depended on an extremely specific scenario (and were dubious even there) and don't impact on even slightly altered scenarios, much less the whole of physics.

Although the fact Bohr's argument was unjustifiable does not prove Einstein was right, I think he probably was – an opinion backed up by [1]. But whether he was or not, since no claim has been made that quantum mechanics asserts what Bohr and Einstein thought it asserted, this whole dialogue is ultimately irrelevant.

3 Result, proof, and remarks

Theorem 3 (Main result) *Let $0 \le \delta < 1$. Suppose a wavefunction Ψ has the property that the probability that an energy measurement would yield an answer lying inside an energy interval of width ΔE, is $\ge 1-\delta$. Suppose the wavefunction evolves in time according to Schrödinger's equation with some time-independent Hamiltonian for a time Δt, resulting in a time-evolved wavefunction Φ. Then*

$$|\langle \Phi|\Psi\rangle| + \delta \ \ge\ (1-\delta)\cos\frac{\Delta t\,\Delta E}{2\hbar}. \tag{7}$$

Proof. WLOG we assume Ψ is a linear combination of orthonormal energy eigenstates with coefficients c_E. Thus if I is the ΔE wide energy interval, the assumption is that

$$\sum\!\!\!\!\!\!\int_{E\in I} |c_E|^2 \ \ge\ 1-\delta. \tag{8}$$

Under Schrödinger time evolution, each energy eigenstate evolves in time by multiplication of c_E by the phase factor $\exp(-iEt/\hbar)$. The maximum difference in angle between any phase factor in I and the phase factor corresponding to the midpoint of I, is then $\theta = (\Delta E)t/(2\hbar)$. Hence the portion of the inner product arising from I is $\ge (1-\delta)\cos\theta$. The remaining part of the inner product is $\le \delta$ in absolute value – the worst that could happen is that the eigenstates outside I get rotated fully 180° away. Then the inner product obeys (EQ 7), QED.

Remarks.

(i) In other words, if you want the time-evolved wave function to differ substantially from the original one ("differ" if the sense of having correlation smaller than some constant, i.e. "angle" between the two wavefunctions as Hilbert space vectors larger than some constant) you need to wait for a time Δt at least of order $\hbar/\Delta E$. Application to, e.g. state transitions or radiative decay is immediate. For example in a radioactive β-decay we go from a state in which the β particle is highly probably inside the nucleus, to a state in which it is highly probably far away. Hence we transition from one state Ψ to another Φ which is very nearly orthogonal to it. Hence, if we are 90% certain that the energy emitted will lie within some ΔE-wide interval, then the time Δt required for the decay must obey $.1 \ge .9\cos([\Delta t\,\Delta E]/[2\hbar]$, so that $\Delta t\,\Delta E \ge 2\hbar \arccos(1/9) > 2.9189\hbar$.

(ii) Let $X = \overline{E - E_0} = \langle\Psi|H - E_0|\Psi\rangle$ be the expectation value of the energy above the ground state energy E_0. (The theorem that a ground state exists for any system with Coulombic potentials, was surveyed in [21]; see also lemma 1 of [16] combined with lemma 4 of Kato's [15] and its improvement proposition 1 of [17] for $L^2 + L^\infty$ interparticle potentials.) Then for any $k = \delta^{-1} > 1$, the probability that the energy lies in an interval $[E_0, E_0 + Xk]$, is $\ge 1 - 1/k$ by Markov's inequality ([13] 26.1.36). Hence an immediate corollary is that

$$k^{-1} + |\langle\Phi|\Psi\rangle| \ \ge\ (1-k^{-1})\cos\frac{\Delta t\, k\,(\overline{E-E_0})}{2\hbar}, \quad \text{for any } k>1. \tag{9}$$

In other words, it is impossible for a wavefunction to evolve to a state sufficiently different that it is at (say) a $\ge 45°$ angle away, unless the time lapse exceeds

$$\Delta t_{45°} \ \ge\ 0.045845\,\hbar\,/\,(\overline{E-E_0}). \tag{10}$$

More generally to evolve an angle θ away, one must wait

$$\Delta t_\theta \ \ge\ \frac{2\hbar}{(\overline{E-E_0})k}\arccos\frac{k\cos\theta + 1}{k-1}, \tag{11}$$

for some $k \ge 2/(1-\cos\theta)$. (EQ 10) came from using $\theta = 45°$ in (EQ 11) with the optimal value $k \approx 10.1215$.

(iii) Similarly by taking $k = 2.84245$ when $\theta = 90°$ we conclude: to evolve to an orthogonal state requires waiting at least a time

$$\Delta t_{90°} \ \ge\ 0.35078253\,\hbar\,/\,(\overline{E-E_0}). \tag{12}$$

[7] For a typical rendition, see [26]. Omnes concludes that of course Bohr was right and in fact regards the proof by Thirring as a later justification, with the caveat that, presumably due to a weakness in his mathematical argument, Thirring did not manage to get the right coefficient 1/2 of $\hbar$. Well, no: Thirring's result does not assert what Omnes thought – it says nothing about uncertainty in a time measurement. Also, the tightness of the numerical coefficients in theorems 2 and 3 indicates that 1/2 is not "right," at least in these formulations.

See [24] for a related claim.
(iv) We can also get a statement[8] about energy *variance* by applying Chebyshev's inequality ([13] 26.1.37) to see

$$k^{-2} + |\langle\Phi|\Psi\rangle| \ \geq \ (1-k^{-2}) \cos\frac{k\,\Delta t\,\Delta E}{\hbar}, \quad \text{for any } k \geq 1, \tag{13}$$

where now ΔE denotes the square root of the variance of E. Then to evolve to a state at least angle θ away requires

$$\Delta t_\theta \ \geq \ \frac{\hbar}{k^2\,\Delta E} \arccos\frac{k^2\cos\theta - 1}{k^2-1}, \tag{14}$$

for some k with $k^2 > 2/(1-\cos\theta)$.
(v) Since the math is the same (but what we were calling k we now are calling k^2, and ΔE is redefined) we see that (EQ 10) and (EQ 12) still hold with ΔE redefined to be the square root of the variance of E.
(vi) Inequality (EQ 7) is tight: consider two energy eigenmodes at the interval endpoints, each with amplitude $1/\sqrt{2}$, with $\delta \to 0^+$.
(vi) An attempt to get an energy-time uncertainty principle based on a time/frequency Fourier analysis would seem unable to yield anything interesting if E never changes (which it can't, for the universe as a whole). However, (EQ 7) is still applicable and interesting even in that case.

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