Applications of geometric separator theorems

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Abstract — **The companion paper "Geometric separator theorems" proved a large number of separator theorems about geometrical objects. We now find geometric separator theorems about geometrical *graphs*, and applications (mostly algorithmic) of them.**

Examples: I: There exists a rectangle crossed by the minimal spanning tree of N sites in the plane $\leq (4\cdot 3^{1/4}+o(1))\sqrt{N}$ times, having $\leq 2N/3$ sites inside and outside. II: Similar results for other geometrical graphs such as minimum matchings and traveling salesman tours, and in arbitrary dimensions. III: E-edge 2-connected torus (or Klein bottle) graphs, each of whose valencies and face sizes are ≥ 3, have a 1/3-2/3 separator with $\leq (4+o(1))\sqrt{E}$ vertices in the separator. The separator corresponds to a simple closed curve. IV: We exhibit V-vertex planar graphs such that any 1/3-2/3 separator must have cardinality$\geq\sqrt{5V/2}$. V: We present the first subexponential algorithms for optimal traveling salesman tour and rectilinear Steiner tree of N points in d-space, d fixed. The runtime if $d\geq 2$ is fixed is $N^{O(N^{1-1/d})}$. The same runtime bound applies for finding a $1+O(N^{-p})$ times optimal length Steiner tree in the Euclidean norm if $p>0$ is fixed. VI: Algorithm to determine all intersection relationships among N convex d-bodies with aspect ratios bounded by B. If a point of space can be in at most κ of the bodies (i.e. they are "κ-thick"), then the runtime is $(\log N+\kappa B^d C)d^{O(d)}N$ and the memory requirement is $d^{O(d)}N$, where C is the amount of time required to test if two given objects intersect. VII: Solve linear systems whose graph structure is an intersection graph as in VI, in $O([d\kappa^{1/d}BN^{1-1/d}]^\omega)$ time, $\omega<2.376$. VIII: New data structure with $O(\log N)$ query time, $O(N)$ space consumption, and $O(N\log N)$ build time, to support point location among N convex boundedly thick convex bodies of bounded aspect ratios in bounded dimension. IX: New algorithms for obstacle avoiding short paths among boundedly thick obstacles of bounded aspect ratio in the plane. E.g. this may be solved in preprocessing time $O(N^{3/2}\log N)$ with storage $O(N\log N)$ and query time $O(\log N)$ to report a path 3.01 times longer than optimal (in the L_1 obstacle avoiding metric) between any two given points.

4 Introduction

IN THE COMPANION paper[1] "Geometric separator theorems," we proved a large number of separator theorems about geometrical objects. For example:

Theorem 37 *Given N interior-disjoint squares in the plane, there exists a rectangle (both the squares and the rectangle have sides oriented parallel to the coordinate axes) such that $\leq 2N/3$ squares's interiors are entirely inside it, $\leq 2N/3$ are entirely outside, and $\leq (4+o(1))\sqrt{N}$ are partly inside and partly outside.*

We'll now use these theorems to[2] (C) prove geometric separator theorems about geometrical graphs, and (D) obtain algorithmic (and other) applications of the separator theorems.

As a simple example of a geometric separator theorem for a geometrical graph:

Theorem 38 *Given N sites in the plane, there exists a rectangle R (with sides at angle 45° to the coordinate axes) such that the E-edge rectilinear Steiner minimal tree (RSMT) of the sites, has $\leq 2E/3$ of its edges entirely inside R, $\leq 2E/3$ of its edges entirely outside, and $\leq (4+o(1))\sqrt{E}$ edges cross the boundary of R.*

Proof. This is an immediate consequence of the previous theorem once you know the "diamond property" (lemma 50) of RSMT edges e: the squares with diagonal e, are interior disjoint. □

As an example of an algorithmic application, it now follows (§6.3) that there is an algorithm to find the RSMT of N sites in the plane in subexponential worst case time $N^{O(\sqrt{N})}$.

Here's a more detailed sketch of ingredients C and D.

C. Separator theorems about geometrical graphs (§5, §6). We use the companion paper's separator theorems about geometrical objects as tools to obtain these.

First, in (§5) we get new proofs and variants of the famed "planar separator theorem" of Lipton and Tarjan, e.g. we have a simple curve separator theorem for Torus graphs.

Second, we have (§6) numerous variants of theorem 38: d dimensions, other separators than boxes, (e.g. circles) and other graphs than RSMT, e.g.: Steiner Minimal tree (SMT), optimal traveling salesman tour (TST), Minimum spanning tree (MST), all-nearest neighbor graph (ANN), minimum matching (MM), "spanners," and "banyans," and L_1-norm versions of all of these. The Gabriel graph (GG) and Delaunay triangulation (DT) are examples of graphs *without* geometric separators (§6.7).

The proof idea is somehow to associate an object (or objects) with each SMT edge, then prove these objects have

[1] We use the same notation and basic definitions as there. A preliminary combined full version of both papers (> 50 pages) is available by `ftp ftp.nj.nec.com`, login as anonymous, password = your email address, `cd pub/wds`, `get geomsep.ps`, `quit`. *NECI, 4 Independence Way, Princeton NJ 08540. †Dept of Mathematics and Statistics, University of Melbourne, Parkville VIC 3052, Australia. Research supported by the ARC.

[2] We number these 'C' and 'D' to continue the companion paper's use of 'A' and 'B' – really there are four ingredients. We continue section, equation, and theorem numbering from where the previous paper left off.

bounded aspect ratios and are disjoint (or merely "κ-thick," "κ-overloaded," or "(λ, κ)-thick;" these are weaker versions of the word "disjoint" defined in the companion paper.) Then by use of our previous separator theorems about objects, we get a separator theorem about SMT (or TST, MM; whatever) edges.

D. Applications of our theorems, especially new algorithms and data structures, are discussed in §7.

1. New algorithm to compute optimal traveling salesman tour of N sites in d-space. Runs in $2^{d^{O(d)}} N^{dN^{1-1/d}}$ steps and consumes $O(Nd)$ space. This simplifies and generalizes a previous 2D algorithm by Smith [26].

2. New algorithm to compute optimal rectilinear Steiner tree of N sites in d-space. Runs in $2^{d^{O(d)}} N^{d^3 N^{1-1/d}}$ steps and consumes $O(Nd)$ space. This simplifies and generalizes a previous 2D algorithm by Smith [28].

3. New algorithm to compute $1+O(N^{-p})$ times optimal length Euclidean Steiner tree for N sites in d-space. Runs in $(N^{1+p}d^d)^{O(d^{5/2}N^{1-1/d})}$ time.

4. New data structure for point location among N disjoint (or κ-thick) d-objects with aspect ratios bounded by B. A data structure requiring $d^{O(d)}N$ storage is constructed in $d^{O(d)}N \log N$ preprocessing time, which supports point location queries in $O(\kappa B^d O(d)^d T + \log N)$ time per query, where it takes time T to test if a point is in one given object.

Also, the data structure may be "dynamized" to allow insertion and deletion of objects (but respecting κ-thickness). In this case, the (amortized) time bound for a query increases by a factor of $\log N$, the space requirement is affected by only a constant factor, and the insertion and deletion times are $d^{O(d)} \log N$.

5. New algorithm for point location among N disjoint (or κ-thick) iso-oriented d-boxes. Uses Storage: $O(Nd)$. Build time: $O(d^2 N \ln(N + 2d - \kappa))$. Query time: $\leq \binom{2d \ln(N+2d-\kappa)}{d-2} O(d + \kappa + \log N)$. ♣ This arises from theorem 27. If d is fixed, our results are the same, up to constant factors, as the previous solution by [10][3]. However, the asymptotic behavior of our worst case query time when d becomes large is a factor roughly $(2e \ln 2)^d$ slower than [10]'s. On the other hand, in the best case in which our separators, by luck, happen to yield 50-50 splits, our typical query time will be roughly $d!$ times faster than [10]'s. When the boxes are very small and far apart, 50-50 splitting is normal, so no "luck" is required. Hence, it is not clear which method is better in practice.

6. New algorithm for obstacle avoiding short paths among boundedly thick obstacles of bounded aspect ratio in the plane. E.g. this may be solved in preprocessing time $O(N^{3/2} \log N)$ with storage $O(N \log N)$ and query time $O(\log N)$ to report a path 3.01 times longer than optimal (in the L_1 obstacle avoiding metric) between any two given points.

7. Algorithm to determine all intersection relationships among N κ-thick d-objects with aspect ratios bounded by B. The runtime is $(\log N + \kappa B^d C) d^{O(d)} N$ and the memory requirement is $d^{O(d)}N$. Here C is the amount of time to test if a point is inside just one specified object.

8. Separator theorems for intersection graphs of κ-thick d-objects with aspect ratios bounded by B, and consequently the ability to c-color such graphs in $(c-1)^{O(d\kappa^{1/d} B N^{1-1/d})}$ time (or prove impossibility), solve sparse linear systems with such graph structure in $O([d\kappa^{1/d} B N^{1-1/d}]^\omega)$ time ($\omega \in (2, 3]$ is the exponent in the runtime of $n \times n$ matrix multiplication; [6] showed $\omega \leq 2.376$), and find "universal" sparse graphs containing all such graphs.

Some of these algorithms and data structures look highly practical.

5 Planar, torus, and Klein bottle graph separator theorems

The famous "Planar separator theorem" of Lipton and Tarjan [18] (elaborated by numerous subsequent authors, e.g. Djidjev, who in [9] improved the constant, and in [8] extended it to hold for graphs of any fixed genus) states

Theorem 39 *Any V-vertex planar graph has a vertex subset , call the "separator," of cardinality $\leq \sqrt{6V}$ whose removal separates the graph into two components, each having $\leq 2V/3$ vertices. Furthermore this separator may be found in $O(V)$ time.*

Surprisingly, it is possible to prove theorems of this sort by entirely geometrical methods.

5.1 Circles

Spielman and Teng [29] observed that the Koebe circle theorem [17] states that any V-vertex planar graph may be embedded as V interior disjoint discs in the plane, such that two discs are tangent if and only if the two corresponding vertices were adjacent in the graph. The Koebe circle theorem was re-proven algorithmically by Smith [27] and Mohar [20], who gave polynomial(P, V) time algorithms to find coordinates accurate to P decimal places.

Hence from the 2D case of the Miller-Thurston sphere separator theorems above, we immediately obtain a (polytime) $(3/4, 1.83973\sqrt{V})$ separator theorem for planar graphs [29], and a $(3/4, 1.90463\sqrt{V})$ weighted separator theorem with arbitrary weights (summing to 1) on each vertex, edge, and/or face of a polyhedral planar graph. If the graph is maximal planar, the separator produced will automatically be a simple cycle.

5.2 Squares

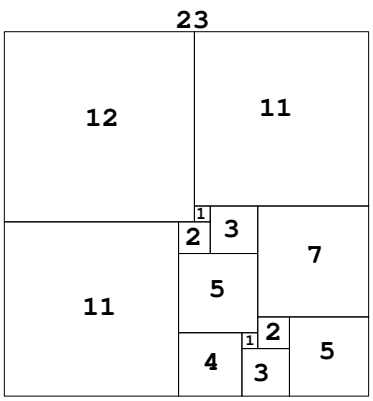
There is a beautiful 1-1 correspondence, first discovered by Brooks, Smith, Stone, Tutte [4], between

- polyhedral (that is, 3-connected) planar graphs G with E edges, one of them distinguished, and
- "squared rectangles" R (that is, a tiling of a rectangle by squares, not necessarily of equal sizes) with $E-1$ tiles.

[3] As extended [26] to work with the disjointness requirement weakened to a κ-thickness requirement.

We'll now describe and prove this.

Each vertex of G will correspond to a maximal horizontal line segment in R (note: such a line segment could be the concatenation of more than one tile side). Each edge of G will correspond to a square tile in R, except for the distinguished edge, which corresponds to the exterior of R. Finally, each face of G corresponds to a maximal vertical line segment in R.



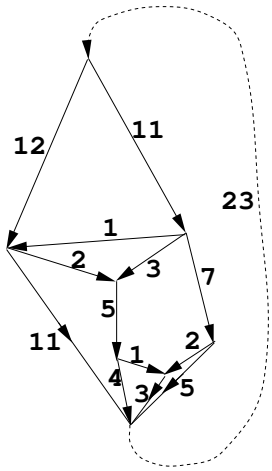


Figure 3: Squared square with 13 tiles. (Bouwkamp 1946.)

Figure 4: Equivalent electrical network with currents shown. Dashed edge is the "distinguished" edge containing the battery; other edges are 1Ω resistors.

Now view each edge of G as a 1-ohm resistor, except for the distinguished edge, which is a battery. The current through each edge will correspond to the width of the corresponding square tile, and the voltage across each resistor will correspond to the height of the corresponding square tile. The currents that will then arise satisfy "Kirchoff's laws," which state that

- **charge conservation** The current flowing into a vertex equals the current flowing out – this corresponds to the fact that the sum of the sidelengths of the squares above some horizontal line segment must equal the sum of the sidelengths of the squares below.
- **no flow cycles** The voltage differences around any cycle (in particular a face) of G, excluding cycles which involve the battery, is 0 – this corresponds to the fact that the sum of the sidelengths of the squares to the left of some vertical line segment must equal the sum of those to the right[4].
- **Ohm law** The current through a 1-ohm resistor is the same as the voltage across it. This corresponds to the fact that the height of a square is the same as its width, i.e. the tiles are indeed squares.

Since any such electrical circuit has a unique valid current flow through it, and as we've seen the equations of current flow are exactly the conditions for validity of a squared rectangle, we see that for each G with a distinguished edge there exists a unique squared rectangle. On the other hand, for each squared rectangle there clearly exists a unique corresponding G.

Now applying theorem 25 instantly shows that for any embedded 3-connected planar graph G with E edges, there exists a Jordan curve (which may travel through faces, or encounter vertices, or cross or traverse edges, of G) crossing or traversing $\leq (4+o(1))\sqrt{E}$ edges and having a total number of encounters with vertices, plus having a total number of trips through faces, totalling $\leq (4+o(1))\sqrt{E}$, and such that this Jordan curve will split any of a variety of weight measures 1/3-2/3 or better.

We conclude

Theorem 40 *For any polyhedral graph with E edges, there exists a smooth closed curve traversing and/or crossing $\leq (4+o(1))\sqrt{E}$ edges in total, whose removal will subdivide the graph into 3 parts, the two main ones each having weight $\leq 2/3$ (for any of a large variety of permissible non-negative weight functions summing to 1, e.g. weights on edges).*

Remarks:

(i) The proof above leads to a polynomial time algorithm, because the square sizes may be found by solving a system of V sparse linear equations embodying Kirchoff's laws. But this algorithm is not particularly fast – $O(V^{1.188})$ time (§7.8).

(ii) The squares proof above was actually only for 3-connected planar graphs ("planar nets") but generalizes to show that 2-connected *torus* graphs, each of whose valencies and face sizes are ≥ 3, have a 1/3-2/3 separator theorem with $\leq (4+o(1))\sqrt{E}$ vertices in the separator via "squared rectangular toruses" (a "rectangular torus" is a rectangle with opposite edges identified to get "periodic boundary conditions."

Squared tori arise[5] as follows. Take a 2-connected embedded torus graph G (each of whose valencies and face sizes are ≥ 3, and whose edges are to be regarded as 1-ohm resistors) and draw a smooth closed non-self-intersecting curve J which cannot be shrunk to a point (due to the fact that it winds exactly once around the "handle" or the "hole" of the torus) and which cuts edges at most once per edge. Wherever J crosses an edge, put a battery in series with the 1-ohm resistor corresponding to that edge. All the batteries have the same voltage and all their polarities are oriented "outward" with respect to J.

(v) This is the first separator theorem for Torus and Klein bottle graphs in which the separator has simple structure, i.e. corresponds to a single closed curve.

[4] Or equivalently, this is charge conservation in the planar dual graph G' using planar dual currents; planar duality corresponds in the rectangle world to turning your head 90 degrees.

[5] One can also make "squared cylinders" [2], which arise very similarly to the special case of squared rectangles except that instead of having a distinguished *edge*, we have a distinguished pair of vertices. Of course, these still are merely a subset of squared tori.
"Squared Moebius strips" and "squared Klein bottles" can also be done (J in the torus construction now becomes a cut across the Moebius strip, respectively across the orientation reversing side of the Klein bottle square, rather than a closed curve).

5.3 *Two lower bounds (one new) for planar separator constants*

A lower bound by Djidjev [9], whose proof we will simplify greatly, is

Theorem 41 (Djidjev) *There exist V-vertex planar graphs such that any subset C of the vertices whose removal splits the graph into A, B with $|A| \le xV$, $|B| \le (1-x)V$, $x \le 1/2$, must have cardinality at least*

$$(1+o(1))\sqrt{2\sqrt{3}\pi x(1-x)V}. \tag{31}$$

Proof. There are configurations of V points on the unit sphere such that their convex hull graph has every edge length (in angular measure) $< \pi\sqrt{2/(\sqrt{3}V)}$, and such that every spherically convex region of area $4\pi x$, $0 < x < 1/2$, x fixed, contains $\le xV\pi/\sqrt{12} + o(V)$ points. (Such configurations may be constructed with, e.g., the techniques of [30]; one approximates the sphere by a sequence of developable surfaces on which are drawn equilateral triangle grids.) Now the result follows from the isoperimetric theorem on the sphere. □

But, as we will now see, Djidjev's lower bound is improvable[6] in the range $0.08112 \le x \le 0.35759$.

The graphs are the "R-refined icosahedra" maximal planar graphs I_R. These are obtained by starting with an icosahedron I_0 and getting I_{R+1} from I_R by dividing every triangle into 4 triangles by adding extra vertices at the edge midpoints) has $20\cdot 4^R$ triangles and $10\cdot 4^R+2$ vertices. All the vertices have valence 6 except for 12 vertices of valence 5. .

Theorem 42 ("Icosahedral" lower bound for planar separator theorem) *Let C be a cycle of I_R with at least $V/2$ vertices outside it ($V = 10\cdot 4^R+2$, R sufficiently large) and at least xV inside. Let $|C|$ denote its length (number of edges).*

- *If $.325 \le x \le 1/2$, then $|C|$ is at least $\sim 5\sqrt{V/10}$.*
- *If $.3 \le x \le .325$, then $|C| \gtrsim \sqrt{(4x+1.2)V}$.*
- *If $\frac{1}{6} \le x \le .3$, then $|C| \gtrsim \sqrt{(6x+.6)V}$.*
- *If $\frac{2}{15} \le x \le \frac{1}{6}$, then $|C| \gtrsim \sqrt{(8x+\frac{4}{15})V}$.*
- *If $0 \le x \le \frac{2}{15}$, at least $\sim \sqrt{10xV}$ edges are required.*

Proof sketch. The proof involves a fairly long case analysis, hence is omitted. But the key ideas are as follows. We claim that for any simple cycle C in I_R of bounded length and containing the maximum number of triangles inside:

1. In an anticlockwise traversal of C (so that the interior is on the left), at each vertex we must either "go straight" or make a "left turn." In this statement, we may define "go straight" to mean that there are three triangles interior to C meeting the vertex, whilst at a left turn there are at most two.
2. It is impossible for C to have 3 consecutive vertices at which we respectively go straight, turn left by an acuter than minimal bend, and go straight, unless C has ≤ 9 arcs.
3. If three consecutive left turns are at u, v and w then the number of steps from u to v and the number of steps from v to w differ by at most 2 (otherwise changing the number of steps between turns by 1 or 2 creates another cycle with the same length and a larger number of triangles inside),
4. The number of left turns is equal to 6 minus the number of 5-valent original icosahedral vertices which are interior to C.

If u is the (approximate) number of steps between left turns, then the length of C is $ku + O(1)$ where k is the number of turns, $k \ge 1$. The number of 5-valent vertices inside is $6-k$.

This reduces the problem to a small finite case analysis, namely, the cases are $k \in \{1,2,3,4,5,6\}$. Also, using more than 1 cycle won't help because the number of enclosed triangles is a concave-$\cup$ function of u. □

Probably these bounds could be improved further by using more complicated families of graphs than I_R, e.g., using refinements of the regular dodecahedron with each pentagonal face subdivided by 5 "spokes." That might extend the region of superiority versus Djidjev's bound further toward $x = 1/2$. But great perseverance would be required to take on the requisite more complicated case analyses.

6 Separator theorems for geometical graphs

6.1 *Definitions of geometrical graphs*

Given N point sites in Euclidean space, a "geometrical graph" is a set of line segments ("edges") whose endpoints include the sites. We'll assume the sites are in general position so that we may avoid worrying about, e.g., nonunique minimum spaning trees.

We've previously ('C' discussion before §1.2) defined TST, SMT, RSMT, MST, MM, ANN, RANN (The 'R' means "rectilinear"), and DT.

The *Gabriel graph* (GG) is the graph of intersite line segments such that the circumballs of the line segments are empty of sites.

It is known [22] that

$$ANN \subseteq MST \subset GG \subset DT. \tag{32}$$

The full paper includes disproofs of various forms of the "Ganley conjecture" [12] that good line separators exist for

[6] For example when $x = 1/3$, we need at least $\sqrt{5V/2} > 1.5811\sqrt{V}$ vertices in the separator, whereas Djidjev's bound was ≈ 1.5551; when $x = 1/4$, we need at least $\sqrt{2.1V} > 1.4491\sqrt{V}$, versus Djidjev's ≈ 1.4284.

RSMT, MST, SMT, RANN, ANN, TST. The disproofs involve exponential spirals and a new analysis techique called "electrical shorting."

6.2 *Minimum spanning trees, Steiner trees, and All nearest neighbor graph*

We'll now argue that MST, SMT, and ANN in the Euclidean metric enjoy geometric separator theorems. This can be (and will be) shown in two ways:

1. We construct a convex object (called a "diamond") of bounded DW aspect ratio enclosing each MST edge, and show that all the diamonds are interior disjoint. Then we apply a previous geometric separator theorem for interior disjoint convex objects of bounded DW aspect ratio.

2. We show that the circumballs of the MST edges, although not necessarily disjoint, are "$2^{O(d)}$-thick" (definition 5). Then we apply the Miller-Thurston separator theorems 32, 33 for κ-thick balls.

As stated, these two approaches only prove a separator theorem for MST, but similar results immediately follow for SMT (since an SMT is the MST of its N sites and its $\leq N-2$ Steiner points) and ANN (since ANN is a subgraph of MST, and hence any κ-thickness and disjointness properties for objects associated with MST edges, are even more true for ANN edges).

The first method of proof is conceptually simpler and shorter; and also it yields the best constants in low dimensions. The second method seems to lead to better constants (by a factor of order d^2) in high dimensions, and is the progenitor of the more powerful techniques we will need later.

Definition 43 *The "diamond" of an MST edge AB denotes [14] the intersection of two cones of half angle 30° with respective apices at A and B, each with axial line AB.*

For example in 2D this is a 60-120-60-120 rhombus whose 60° corners are at A and B; In 3D, the diamonds are the bodies of revolution obtained by rotation of the rhombi.

Lemma 44 (Diamond property for euclidean MSTs) *The diamonds of MST edges in any Euclidean d-space, are interior disjoint.*

Proof. In 2D, this is shown in §8.6 of Gilbert & Pollak [14]. On page 22 of the same paper, a proof in dimensions $d \geq 3$ is presented, and attributed to R.L.Graham and J.H.Van Lint. □

By theorem 25 (in the form of remark (ii) after the proof) applied to 60° rhombi[7] in 2D, we get

Theorem 45 (2D MST separator theorem) *Consider the MST of N sites in the Euclidean plane. There exists a rectangle R such that at most $2N/3$ of the sites (or the MST edges, or at most $2/3$ of any of a wide class of "weights") are wholy inside R, at most $2N/3$ are wholy outside, and $\leq (4\cdot 3^{1/4}+o(1))\sqrt{N}$ MST edges are partly inside and partly outside R.*

In general dimension d, we also get a $2/3$-separating rectangle, but the bound on the number of edge crossings obtained by this technique is $O(d^{3/2})N^{1-1/d}$, which is not as good a bound as we will obtain in theorem 49 below (albeit at the cost of weakening the split balance from $2:1$ to $d+1:1$).

Definition 46 *Let τ_d be the "kissing number" in d-space, that is, the maximum number of interior-disjoint unit d-balls which can touch one.*

It is known [5] that $\tau_1 = 2$, $\tau_2 = 6$, $\tau_3 = 12$, $\tau_8 = 240$, and $\tau_{24} = 196560$; and $\tau_d \leq 2^{0.401d+o(d)}$.

Lemma 47 (3D ball lemma) *We claim that if two 3D balls B_1 and B_2 exist, both containing the origin, and with the radius of B_2 being at least 2 times as large as the radius of B_1, and if two antipodal points of B_1 both lie outside (or on the surface of) B_2, then the angle subtended by the centers of B_1 and B_2 at the origin is at least 60°.* ♣

Theorem 48 (MST circumballs aren't too thick) *The circumballs of the edges of an MST in d-space are M-thick with*

$$M \leq 1 + 2(\frac{4d}{d-1})^d(\tau_d - 1) \leq 2^{2.401d+o(d)}. \qquad (33)$$

Proof. First, we remark that MST edges obey the "empty lune property" that the intersection ("lune") of the two d-balls of radius L centered at A and B (for some MST edge AB of length L) is empty of sites. The circumball of AB is entirely contained in the lune of AB, hence all such circumballs are empty. (Proof: if a site C were in the lune of AB, then C is connected by an MST path to either A or B (first), say WLOG A, and then removing AB and substituting BC would make the MST shorter but still connected, a contradiction.)

Now suppose some point of d-space (WLOG, the origin) is contained in M circumballs. WLOG let the smallest of these balls have diameter 1 and indeed let the diameter of the ith smallest ball be $Q[i]$.

All the $2M$ endpoints of the MST edges defining the balls must lie in the ball of radius $Q[M]$ centered at the origin. These MST edges have total length L with $L \geq M$. However[8], the length of an MST of $2M$ points in a d-ball of radius $Q[M]$ is $L < \frac{2^{3-1/d}dQ[M]}{d-1}M^{1-1/d}$. This leads to a contradiction if $Q[M] \leq 2^{-2+1/d}\frac{d-1}{d}M^{1/d}$. Indeed, we have a contradiction if $Q[k]/Q[j] \leq 2^{-2+1/d}\frac{d-1}{d}(k-j)^{1/d}$ for any j,k, $1 \leq j < k \leq M$.

Therefore, it must be the case that $Q[j] \geq 2Q[j - 2(\frac{4d}{d-1})^d]$, i.e. the balls at least double their diameter after every $2(\frac{4d}{d-1})^d$ balls.

Now, consider two balls B_1 and B_2, the second one having a factor of 2 larger radius. We will be interested in the

[7] Whose area is $1/\sqrt{3}$ for a unit length SMT edge, so we may take $\tau = \sqrt{3}$.

[8] By a lemma omitted in this condensed paper.

centers of these 2 balls, the MST endpoints which form a diameter of B_1, and the origin. These 5 points lie in a 3-dimensional subspace. (Generically 5 points define a 4-space, but here the 3 points on the MST edge are collinear, so we get a 3-space.) Hence by lemma 47, their centers must subtend an angle $\geq 60°$ at the origin.

Since the maximum number of vectors, all $\geq 60°$ apart, which can exist is τ_d, we conclude finally that $M \leq 1 + 2(\frac{4d}{d-1})^d(\tau_d - 1)$. □

By applying separator theorem 33 (theorems 25 and 32 could also be applied) we conclude

Theorem 49 (MST, SMT, ANN separator theorem) *Let MST, SMT, and ANN be the minimum spanning tree, minimum Steiner tree, and all nearest neighbor graphs, respectively, of N sites in d-space, $N > d \geq 1$. Then there exists an d-sphere such that at most $(d+1)N/(d+2)$ of the sites are inside it; at most $(d+1)N/(d+2)$ are outside; and at most $cd^{-1/2}N^{1-1/d}(1+o(1))$ of the graph edges cross the sphere. Here $c = 2^{1.401}\sqrt{\pi/2} \approx 3.31$ suffices. Here the "o" applies when both d and N go to ∞; if only one of them does, it should be replaced by an "O."*

6.3 Rectilinear minimum spanning trees, Steiner trees, and All nearest neighbor graph

Rectilinear MSTs, SMTs, and ANN graphs enjoy geometric separator theorems. Some parts of these demonstrations are analagous to our preceding arguments for the non-rectilinear graphs, but other parts need to be different.

The easiest example of an analogy that does work, is for 2D RMSTs, which satisfy a "diamond property" analogous to lemma 44.

Lemma 50 (RMST diamond property in 2D) *If each (horizontal or vertical) RMST line segment in the plane is regarded as the diagonal of a (45° tilted) square, then: all these squares are interior disjoint.*

Proof. WLOG CD is horizontal. If AB is vertical the result is immediate because CD's diamond is empty of A and of B. (It's impossible for any other site to be inside an RMST edge's L_1 circumball.) So AB is horizontal too; say A is left of B, C is left of D WLOG. Let AB be shorter than CD WLOG. (Equal lengths are possible too, but we ignore that case by a random infinitesimal pre-perturbation.) If the diamonds corresponding to AB and CD overlap, where AB is shorter than CD, then we claim $\ell(CA) < \ell(CD)$ and $\ell(DB) < \ell(CD)$, a contradiction with the claim that AB and CD are both RMST edges. To see that, start at the corner of CD's diamond contained in AB's, WLOG this is the uppermost corner of CD's, CD is horizontal, and AB is horizontal. This corner is closer to both of $\{A, B\}$ than it is to either of $\{C, D\}$. Now walking left to A only makes you get closer to C. Similarly walking right to B only makes you get closer to D. □

Definition 51 *The "θ-diamond" of a line segment AB means the intersection of the cone with apex A and axial ray $\vec{AB}$ with the cone with apex B and axial ray $\vec{BA}$, where both cones have halfangle θ.*

As an example of an analogy that does not work:

Lemma 52 *RMST and RANN do not enjoy any disjoint θ-diamond property, even in 3D and even for arbitrarily small $\theta > 0$.*

Proof of lemma 52. In 2D draw two length-1 edges crossing one another at right angles at their midpoints. Now join the Eastmost and Southmost vertices by a long chain of tiny RMST edges that stay far away from the crossing. Now lift the "crossing" out of the plane by any $\epsilon > 0$, and we claim the result (figure 5), is a valid RMST, and even with care a valid rectilinear all nearest neighbor graph, in 3D. □

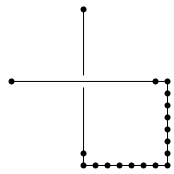


Figure 5: A 3D rectilinear all nearest neighbor graph with only infinitesimally skinny disjoint diamonds.

But RSMT still does enjoy a diamond property, lemma 54 below.

Lemma 53 (Empty octahedron property) *The L_1 ball whose diameter is an RMST (axis-aligned) edge, is empty of sites.*

Proof. Same as proof of the empty lune property mentioned in the proof of theorem 48, only now with L_1 distances. □

Lemma 54 (RSMT diamond property) *If $d \geq 3$ and $\theta < \operatorname{arccot}(2\sqrt{d-1})$, then the θ-diamonds of the line segments of an RSMT in d-space are interior disjoint.* ♣

When $d = 2$, we may combine theorem 25 and lemma 50 to get

Theorem 55 (RMST, RSMT, RANN separator theorem when $d = 2$) *Let MST, SMT, and ANN be the minimum spanning tree, minimum Steiner tree, and all nearest neighbor graphs, respectively, of N sites in 2-space, $N > 2$. Suppose the number of edges in such a graph is E. Then there exists a rectangle, rotated 45° to the coordinate axes, such that at most $2N/3$ of the sites are inside it; at most $2N/3$ are outside; and at most $4(1 + o(1))\sqrt{E}$ of the graph edges cross the box boundary.*

The full paper gives d-space versions of RSMT separator theorem and proves a $d^{O(d)}$ bound on the thickness of RMST L_1 circumballs, showing the existence of a 1/3-2/3 box-separator for RMSTs in d-space with $O(d^2E^{1-1/d})$ of the E RMST edges crossing the box boundary.

6.4 Optimal traveling salesman tours

Lemma 56 *Let $\lambda > 1$. Then there exists κ_λ such that the set of circumballs of the edges of every minimal TST is $(\lambda, \kappa_\lambda)$-thick. (See definition 6.) Indeed it will suffice if*

$$\kappa_\lambda = 16^d \log_2 \lambda, \tag{34}$$

and it will also suffice if the TST is merely "2-optimal" (i.e. can't be shortened by removing 2 edges and substituting 2 others).

Proof. Let $\epsilon > 0$ be fixed. We can choose κ large enough that if some point is covered by more than κ circumballs whose sizes vary by factor at most λ, then by the *pigeonhole principle*, at least three of these circumballs come from edges aa', bb' and cc' whose lengths (after scaling) are all within ϵ of 1, and such that a, b and c lie in a ball of radius ϵ, as do a', b' and c'.

The minimal TST, outside these three edges, joins them in a cyclic fashion, and clearly at least one of these joins must be from a, b or c to a', b' or c'. Assume wlog that it is a to b'. Then replacing the tour edges aa' and bb' by ab and $a'b'$ shortens the tour provided ϵ was chosen sufficiently small – a contradiction.

In fact it will suffice in the first paragraph if

$$\kappa = \frac{\log \lambda}{\log([1+\epsilon]/[1-\epsilon])} \left(\frac{1+\epsilon}{\epsilon}\right)^{2d} \tag{35}$$

and it will suffice in the second paragraph if $\epsilon = 1/3$, which leads to (EQ 34). □

To apply part (c) of theorem 25, we also need to know that at most a constant number of TST edges larger than the rectangular "separating annulus" in the proof of theorem 25, can intersect it. (A very large number of TST circumballs could intersect the rectangle, but we ignore them unless the TST edge itself does.)

Lemma 57 *At most a constant number ($2^{O(d)}$ suffices) of TST edges of length ≥ 1 can intersect a unit cube or ball. (It will suffice if the TST is 2-optimal.)*

Proof. Suppose not. Then among the unboundedly large number of such TST edges, we may by the *pigeonhole principle* find 3 all at angles within ϵ of each other, and "nearly overlapping," i.e. a length of at least 1/2, say, of one is very close to a similar length of the other. But then the "minimal" TST could be shortened. □

We may now combine lemmas 56 and 57 and part (c) of theorem 25 to get

Theorem 58 (TST separator theorem) *Let TST be any 2-optimal traveling salesman tour of N sites in d-space, $N > d \geq 1$. Then there exists an iso-oriented d-box such that at most $2N/3$ of the sites are inside it; at most $2N/3$ are outside; and at most $O(d)N^{1-1/d} + 2^{O(d)}$ of the TST edges cross the boundary of the box.*

6.5 Minimum matching

An argument very similar to (but easier than) the TST argument in §6.4 shows that

Theorem 59 (MM separator theorem) *Let MM be the min-length matching of N sites in d-space, $N > d \geq 1$, N even. Then there exists an iso-oriented d-box such that at most $2N/3$ of the sites are inside it; at most $2N/3$ are outside; and at most $O(d)N^{1-1/d} + 2^{O(d)}$ of the MM edges cross the boundary of the box.*

Remark. It is also possible to prove a result like this by showing that the θ-diamonds of MM edges are $2^{O(d)}$-overloaded, if θ is a sufficiently small constant.

Remark. No matter how small $\theta > 0$ is, the θ-diamonds of MM edges can be unboundedly thick as $N \to \infty$ with θ and d fixed. Therefore, no argument based on κ-thickness could have achieved anything for MM (and presumably the same is true for TST).

6.6 "Spanners" and "Banyans"

Definition 60 *A "$(1+\epsilon)$-spanner" of a set of points is a subgraph of the complete Euclidean graph where for any u and v the length of the shortest path from u to v is at most $(1+\epsilon)$ times the Euclidean distance between u and v.*

Arya et al. [1] showed (building on earlier work by [7]) – and their work was redone more quantitatively by in an appendix of [23]) that for N sites in d-space, and any $\epsilon > 0$, $(1+\epsilon)$-spanners exist

1. Whose total length is only $d^{O(d)}$ times longer than the Steiner minimum tree of the N sites
2. With maximum valency $(d/\epsilon)^{O(d)}$,
3. Which may be constructed in time $(d/\epsilon)^{O(d)}N + O(dN \log N)$.

We would now like to indicate here that furthermore, WLOG, these spanners also obey a (λ, κ)-thickness property. Specifically,

Theorem 61 *WLOG, for some $\theta = \Omega(\epsilon)$, the θ-diamonds (definition 51) of the Arya et al. spanner edges are (λ, κ)-thick, for some $\kappa = (d/\epsilon)^{O(d)}$ and with $\lambda = (d/\epsilon)^{O(1)}$.*

Proof sketch. Follow the argument, sketched in Arya et al. [1]'s section 5 and again more quantitatively in the appendix of [23], proving (what [23] call) "(κ, c)-isolation."

This argument remains valid, at the present level of precision, if, instead of small cylinders of height and radius $c\ell$ (where ℓ is the length of the spanner edge), which are κ-thick, we instead use skinny diamonds, of width $c\ell$. □

A similar result necessarily holds for [23]'s "banyans."

Lemma 62 *At most a constant number ($\epsilon^{-O(d)}$ suffices) of Arya spanner edges of length ≥ 1 can intersect a unit cube or ball.*

Proof sketch. Suppose not. Then among the unboundedly large number of such spanner edges, we may by the pigeonhole principle find 2 at angles within ϵ of each other, and "nearly overlapping," i.e. a length of at least 1/2, say, of one is very close to a similar length of the other. But this would contradict the spanner "shortest edge in ϵ-cone" properties enjoyed by the Arya et al [1] construction. □

We may now apply part (c) of theorem 25 to get

Theorem 63 (Spanner & Banyan separator theorem) *Let B be the $(1+\epsilon)$-spanner or $(1+\epsilon)$-banyan graph used in [23]. Then there exists an iso-oriented d-box such that at most $2N/3$ of the sites are inside it; at most $2N/3$ are outside; and of the E edges in the spanner (or banyan), at most $(d/\epsilon)^{O(1)}E^{1-1/d}$ of the edges cross the boundary of the box.*

Theorem 63 has tremendous applicability. For example, it immediately proves separator theorems for the $(1+\epsilon)$-approximately optimal SMTs, MSTs, TSTs, and MMs arising by replacing all edges in the optimal versions, by paths in the spanner or banyan.

6.7 Delaunay triangulations without separating circles

Since we've just demonstrated that a large number of well known geometric graphs have geometric separator theorems, it seems only fair to mention a graph without one – the "Delaunay triangulation."

For N sites in 3D, the Delaunay triangulation edge-graph can be the complete graph K_N [25] so clearly there is no separator, even a graph-theoretic one.

For N sites in 2D, the Delaunay triangulation edge-graph is planar and hence always has a *graph-theoretic* $(2/3, \sqrt{6N})$ separator (cf. §5), but we argue that it need not have any separating *circle*. To do so, we exploit the fact [11] that the Delaunay triangulation of N sites in the plane, is the same as the convex hull of those N sites, stereographically projected onto a sphere in 3D. The separating circle is then going to correspond to a plane cutting the sphere and the convex hull.

Theorem 64 *Let $c > 0$. Then there exist N point sites on a sphere in $\mathbf{R}^3$ such that any plane with at least cN sites on each side of it, must cross at least $\Omega(N)$ edges of the convex hull of the sites.*

Proof sketch. Consider some large but constant number K of roughly equal spherical caps packed on the surface of the sphere. Space $(N/K - 1)$ sites uniformly around the perimeter of each cap, and one site at each cap center. Notice the convex hull graph will include as subgraphs all the "spoked wheels" corresponding to each of these caps. If a plane cuts a constant fraction of the perimeter off any cap, then it must cross at least $\Omega(N/K)$ convex hull edges, proving the theorem – unless it cuts *precisely* through the cap center. But if the caps are in "general position" then it is impossible for a plane to cut precisely through the center of more than 3 caps. Also, if the caps are packed reasonably densely and K is large enough as a function of c (It will suffice if $K = \Omega(1/c)$) then it is impossible to avoid cutting at least a constant fraction off at least a constant number (in fact $\Omega(\sqrt{K})$) of caps[9]. □

Remark. The same construction also shows that the 2D Gabriel graph GG need not have a circle $o(N)$-separator.

[9] This may be seen by considering cutting the usual hexagonal penny packing by a line – e.g. no matter what line one uses, a constant fraction will be cut off a constant fraction of the cut pennies.

7 Applications: Algorithms and data structures

7.1 Point location in κ-thick d-objects with bounded aspect ratios

Given: A set $\mathcal{S}$ of κ-thick objects with CV aspect ratios (definition 2) bounded by B.
Task: To preprocess them to create a data structure to support "point location queries."
These queries are: "Name the objects containing the point $\vec{x}$."

Theorem 65 *This problem may be solved with a data structure that may be stored in $d^{O(d)}N$ memory locations. The expected time required to build it (the build algorithm is randomized) is $d^{O(d)}N\log N$. Point location queries require $O(\kappa B^d O(d)^d C + \log N)$ steps. Here C denotes the time to determine if a point is in one given object.*

The data structure may be "dynamized" to allow also the insertion and deletion of objects. In this case, the (amortized) time bound for a query increases by a factor of $\log N$, the space requirement is affected by only a constant factor, and the insertion and deletion times are $O(1/N)$ times the build time of a static data structure for N objects.

Proof.

If $N \le \kappa B^d O(d)^d$ do not build a data structure – we'll just do point location by brute force.

Otherwise, use the appropriate variant of the separator theorem 25 (theorems 32 and 33 could also be used, in some cases) to decompose the boxes recursively into a binary tree. The two child subtrees of a node correspond to the objects (1) inside of or overlapping, (2) outside of or overlapping the separator.

Because each of the 2 sets is guaranteed to be a constant factor smaller, the tree has depth $O(\log N)$ (for theorem 25; $O(d\log N)$ for theorem 32). The 2 children of a node may in total correspond to more objects than were known to its parent, due to duplication of objects overlapping the separator. However, it is easily verified[10] that the total storage amplification factor remains bounded by a constant independent of N, in fact $d^{O(d)}$. Then because each non-leaf node of the tree has outvalency 2 and requires $O(1)$ storage (2 pointers, a few numbers describing the separator shape) and the leaf nodes store disjoint sets of objects, the total storage required is only a $d^{O(d)}$ factor larger (at most) that the storage required to write down the input objects in the first place.

To locate a point: If the root is childless, determine the set of boxes the point is in by brute force examination of all the boxes (of which there will be $\kappa B^d O(d)^d$).

Otherwise, see if the point is inside, outside, or on the separator boundary for the root node. If inside or on, explore subtree (1) recursively. Otherwise, explore subtree (2) recursively.

The total query time will be $O(\kappa B^d O(d)^d C + \log N)$ (with theorem 25; with theorem 32, where the objects must be balls, it is $O(\kappa 2^{O(d)}C + d\log N)$).

[10] Since $(1 + dN^{-1/d})^{d\log N}$ is bounded by $d^{O(d)}$.

To build the data structure, employ the randomized algorithmic version of theorem 25 in §3.7.2. The expected time to build the data structure is then $d^{O(d)}N\log N$.

Finally, the dynamization results are a standard application of the ideas in [21]. In short, one maintains $O(\log N)$ different static structures, each a factor ≈ 2 larger than the preceding one. Queries are performed by searching all $\log N$ structures. Insertions are performed by destroying and rebuilding the smallest structure – although if the resulting structure would be too large, then a "carry" is performed, and a destruction and rebuild of the next larger structure is required. Deletions are done "lazily" by simply marking deleted objects as "nonexistent." Once 50% of the objects become nonexistent, a global rebuild is performed to get rid of them. □

7.2 Finding all intersections among N κ-thick d-objects with bounded aspect ratios

Given: A set $\mathcal{S}$ of κ-thick objects with CV aspect ratios bounded by B.
Task: To find all intersection relationships among pairs of objects.

Theorem 66 *The problem above may be solved in time* $(\log N + \kappa B^d T)d^{O(d)}N$ *while consuming* $d^{O(d)}N$ *memory locations. Here it takes time* T *to determine whether two given objects intersect.*

Proof. In the point-location tree structure of the previous section, objects can only intersect other objects if they are in the same leaf of the tree. Since each tree leaf corresponds to a set of objects of cardinality $\kappa B^d O(d)^d$ at most, and the total number of objects in leaves, counting all duplicated objects multiply, is $O(N)$, the total runtime (assuming the tree has already been constructed)[11] is $\kappa O(d)^d NT$. □

7.3 Finding the optimal traveling salesman tour of N sites in d-space

Given: N sites in d-space.
Task: To find their optimal traveling salesman tour.

Theorem 67 *This*[12] *may be accomplished in time*

$$2^{d^{O(d)}}N^{O(dN^{1-1/d})}. \tag{36}$$

The algorithm will be to guess the separating box of theorem 58 among $\lesssim [(N+1)N]^d$ inequivalent possibilities. Then the optimal tour must cross this box at $O(dN^{1-1/d}) + 2^{O(d)}$ places out of $\leq (N-1)N$ possibilities. Guess them too. Finally, for each such guess, guess also the "boundary conditions" at the crossing points, that is, how the tour joins the crossing points on each side of the separator surface. For C crossing points, this is $C!$ possible kinds of boundary conditions. (In the plane, there are only $2^{O(C)}$ possibilities [26].)

Finally, solve the two smaller "traveling salesman tour plus boundary conditions" problems inside and on, and outside and on, the separator, recursively. (If $N < d^{O(d)}$, solve the problem by brute force in $2^{O(N)}$ steps.)

Needless to say, all "guessing" must be done exhaustively.

The runtime $T(N)$ obeys the recurrence

$$T(N) \leq [(N+1)N]^d \cdot N^{O(dN^{1-1/d})+2^{O(d)}}[T(2N/3)+T(N/3)]. \tag{37}$$

The solution is

$$T(N) = 2^{d^{O(d)}}N^{O(dN^{1-1/d})}. \tag{38}$$

The space needs can be kept linear [26]. □

7.4 Finding the rectilinear Steiner minimal tree of N sites in d-space

Given: N sites in d-space.
Task: To find their optimal rectilinear Steiner tree.

Theorem 68 *The number of steps needed for the task is*

$$2^{d^{O(d)}}N^{O(d^3N^{1-1/d})}. \tag{39}$$

A "rectilinear Steiner tree" (RST) is a collection of line segments parallel to the coordinate axes, which interconnect N given sites. The rectilinear Steiner *minimal* tree (RSMT) is the shortest such network.

Some properties of RSMTs:

1. They are trees. (If a cycle existed, one could shorten the RSMT by removing the longest edge in the cycle, contradicting minimality.)
2. The angle formed by 2 coterminous line segments is $\geq 90°$.
3. Each vertex has valency $\leq 2d$.
4. The total number of "Steiner points" (points of valency ≥ 3 which are not sites) is $\leq N-1$.
5. The total number of line segments is $< 2dN$.

A RST "topology" is a specification of the tree structure of the RST, indicating where each site lies, but ignoring the locations of 2-valent non-site corners on paths, and ignoring geometric information (such as lengths, coordinates) generally.

A fundamental theorem about RSMTs was shown by Hanan [16] when $d = 2$, but it seems to be only a "folk theorem" for general d (meaning: we were unable to find a proof in previous literature!). This is

Theorem 69 *The shortest RST with a given "topology" WLOG is one in which all line segments are edges (1-flats) in the arrangement of* dN *hyperplanes orthogonal to the coordinate axes going through the sites.*

[11] Note, because we counted all duplicated objects multiply, we have correctly taken account of the fact that an object could be in more than one leaf of the tree.

[12] We assume we are using a real RAM [22] or similar model of computation, so that difficulties arising from having to determine which of two sums of square roots of integers is greater [13] may be ignored.

Proof. In the shortest RST, the coordinates of a Steiner point are determined by the coordinates of its $\leq 2d$ neighbors as follows: the ith coordinate is the median of the ith coordinates of all the neighbors. (The "median" of an even number of values is non-unique, and any value lying at or between the bimedians may be used without affecting the length.)

Hence, the complete system of equations for the optimal coordinates of all Steiner points in a given RST topology are a system of linear equations, in which every equation is of the form $a = b$. Hence, every Steiner point's ith coordinate is the same, WLOG, as some site's ith coordinate. □

This theorem implies that WLOG there are only N^d possible locations for Steiner points (lying on an $N \times N \times \ldots \times N$ "grid") and only $\binom{N}{d-1} d! < dN^{d-1}$ possible lines that RSMT line segments could be subsegments of. Hence we've reduced the problem of finding the RSMT to a finite search problem.

Our algorithm will be to guess the separating d-box of the RSMT separator theorem from among $\lesssim N^{O(d)}$ inequivalent possibilities. In 2D, it is probably best to use a d-box rotated 45° due to theorem 55, but in high dimensions we just use a box aligned with the coordinate directions.

Then the optimal RSMT must cross the separator boundary at $O(d^2 N^{1-1/d})$ places out of $< 2dN^{d-1}$ possibilities. Guess them too. Finally, for each such guess, guess also the "boundary conditions"' at the crossing points, that is, how the RSMT partitions the joinings of the crossing points induced by the RSMT on each side of the separator surface.

For C crossing points, this is certainly $< 4^C C!$ possible kinds of boundary conditions. (In the plane, there are only $2^{O(C)}$ possibilities [26].)

Finally, solve the two smaller "RSMT plus boundary conditions" problems inside and on, and outside and on, the separator, recursively. (If $N < d^{O(d)}$, solve the problem by brute force in $2^{O(N)}$ steps.)

Needless to say, all "guessing" must be done exhaustively.

The runtime $T(N)$ obeys the recurrence

$$T(N) \leq N^{O(d)} (8dN^{d-1})^{d^2 N^{1-1/d}} [T(2N/3) + T(N/3)]. \tag{40}$$

The solution is (we assume $N > d$)

$$T(N) = 2^{d^{O(d)}} N^{O(d^3 N^{1-1/d})} \tag{41}$$

The space needs are linear. □

7.5 *Euclidean Steiner minimal trees in the plane*

Can the exact SMT on N sites in the plane be found in $N^{O(\sqrt{N})}$, or anyhow $2^{o(N)}$, time[13]?

Theorem 70 *Given N sites in d-space, $2 \leq d < N$, and any fixed real $p > 0$. An algorithm running in time*

$$(N^{1+p} d^d)^{O(d^{5/2} N^{1-1/d})}. \tag{42}$$

exists to find a Steiner tree at most $1 + N^{-p}$ times longer than the Steiner minimal tree. ♣

The full paper has numerous other algorithms pertaining to ESMTs, including ways to get exact optimal ESMTs in subexponential time under certain assumptions, which conjecturally hold for random points in a square.

7.6 *Approximate obstacle avoiding shortest paths in the plane*

Given: A set $\mathcal{P}$ of M convex polygonal obstacles in the plane. We will suppose them κ-thick (note: this allows us to make nonconvex obstacles) and they each have DW aspect ratio $\leq B$, and they have the property that the points where the boundaries of a given pair of them intersect, or the points where a given object's boundary intersects a line, may be computed in $O(1)$ time. (This computational property is true for, e.g., triangular and rectangular obstacles.)
Task: To preprocess these polygons to create a data structure to support "approximate shortest path queries."
These queries are: Given two arbitrary points s and t, find an approximately shortest path (avoiding the obstacles) between s and t (or report that there is no such path) and/or just report the length of such a path.

Theorem 71 *Let $\epsilon > 0$. This task may be solved in preprocessing time $O(B\kappa^{1/2}\epsilon^{-1} N^{3/2} \log N)$ with storage $O(B^2\kappa\epsilon^{-2} N \log N)$ (this must be multiplied by $\min(\epsilon^{-1/2}, \log N)$ for L_2 metric) and query time $O(B^4\kappa^2\epsilon^{-2} + \log N + P)$ (the $\log N$ term may be eliminated if s and t are obstacle vertices) and approximation factor $3 + \epsilon$, where $N = O(\kappa BM)$ is the number of boundary segments in the arrangement of the M obstacles, and paths are measured in the L_1 metric. P is the size of the output, i.e. $P = O(1)$ if only the distance is reported, but if the entire path is reported, $P = O(N)$ is the number of segments in that path. In the path reporting case the storage requirement grows to $S(N) = O(B\kappa^{1/2}\epsilon^{-1} N^{3/2})$.* ♣

It seems likely that our geometric separator theorem may not have been needed here, because the plain Lipton-Tarjan theorem could have been made to suffice. But our approach is undoubtably simpler, plus potentially may be extended to higher dimensions in the future.

7.7 *Coloring, independent sets, and counting problems*

For optimally coloring, or finding maximum independent sets in, or finding maximum cliques in (or counting cliques, or maximal cliques, in), intersection graphs of d-objects with bounded aspect ratio and bounded thickness in *fixed* dimension d, $2^{O(N^{1-1/d})}$ time suffices [24]. Counting k-colorings may be accomplished in $(k-1)^{O(N^{1-1/d})}$ time.

7.8 *Gaussian elimination for systems with the graph structure of an intersection graph of d-objects of bounded aspect ratio and thickness*

In a V vertex graph family with a $(\alpha V, \beta V^p)$ separator theorem one may perform "Gaussian elimination" to solve a

[13] This is open, but a positive solution to this problem – albeit dependent on a conjecture – was given in [28].

system of linear equations in time $T(V)$, where $T(V)$ obeys the recurrence

$$T(V) \le T(\alpha V) + T((1-\alpha)V) + (\beta V^p)^\omega. \qquad (43)$$

Here ω, $2 < \omega \le 3$, is the exponent for the runtime of dense matrix multiplication. If $\omega p > 1$, this solves to

$$T(V) = O(\frac{\beta^\omega}{1-\alpha^{p\omega}-(1-\alpha)^{p\omega}} V^{p\omega}). \qquad (44)$$

If we use $\omega \le 2.376$ [6] and $p = 1-1/d$, we get subquadratic time when $d \le 6$. This idea (sans ω) dates to [19].

In contrast, the "conjugate gradient method" [15] may be used to solve any system of V linear equations, where there are E nonzero terms in the matrix, in $O(VE)$ arithmetic operations. For a sparse graph – with $E = O(V)$ – this is quadratic time.

7.9 Universal graphs

Theorem 72 *There exists a V-vertex graph with $O(d\kappa^{1/d}BV^{2-1/d}) + [O(d)^d\kappa B^d]^2$ edges containing all intersection graphs of V convex d-objects with CV aspect ratio bounded by B and thickness κ.*

Proof sketch. The graph is constructed recursively by placing $O(d\kappa^{1/d}BV^{1-1/d})$ vertices in the "middle" and joining each of these middle vertices to every vertex. Then two disjoint sets of $V/2$ vertices each are chosen and one recursively constructs universal graphs on these two subsets. The fact that this graph is universal arises from the fact that our intersection graphs have 1/2-1/2 separators of size $O(d\kappa^{1/d}BV^{1-1/d})$, which arises from recursive application of our 1/3-2/3 geometric separator theorem [18]. The recursions stop, and one uses a complete graph, when $V < O(d)^d\kappa B^d$. □

Theorem 73 *There exists a universal V-vertex graph with $O(d\kappa^{1/d}BV^{2-2/d})$ edges containing all bounded valence intersection graphs of V convex d-objects with CV aspect ratio bounded by B and thickness bounded by κ.*

Proof. Follows from separator theorem 25 (a) and [3]. □

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