The Gamma function revisited

Warren D. Smith*
warren.wds@gmail.com

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Abstract — **We find several new representations of the Gamma function (and related functions such as $R(z) = \Gamma(z)/\Gamma(-z)$, Binet's function $\mu(z)$, and $\ln\Gamma(z)$) as integrals and partial-fraction-like expansions. We also present convergent and/or better versions of Stirling's formula, fully general reflection and shift formulas for the Gamma function, new continued fractions and new error estimates (such as understanding the convergence rate of Stieltjes' CF for the first time), and a new formula for the Beta function. While some of these seem mere curiosities, others appear to yield the best known numerical algorithms for evaluating $\Gamma(z)$.**

1 Preliminaries

We shall often refer to formulas in the *Handbook of Mathematical Functions* [17] by "HOMF" followed by their formula number. Chapter 6 of HOMF is about the Gamma function and if it were being rewritten today, then in our opinion it ought to include many of the results in the present paper.

Because we do not believe in "Littlewood's dictum" that identities need merely to be stated but not proven (supposedly because their proofs are then comparatively trivial), we shall give proofs, but because we *partially* believe in that dictum those proofs usually will be sketchy rather than detailed. The philosophy is that the purpose of our sketchy proof is *not* to convince the reader of correctness – in practice for results of this ilk greater confidence of correctness is got by performing numerical checks than by verifying a proof – but rather to *inform* the reader of how the result arose, and allow him to produce a fully detailed proof should he so choose. Littlewood's dictum is more true in the modern computerized era than it was in Littlewood's day, since symbolic manipulation systems now make it easier to carry out the steps of such a proof sketch in detail.

Our most important new[1] results are the new integral representations EQs 65, 81, 70, 72, 73, the new convergent version of Stirling' formula EQ 38 and the first-ever analysis of the convergence rate of Stieltjes' continued fraction (theorem 2); the "shift by half" trick that improves all varieties of Stirling formula (§4), the reflection, shift, and doubling formulas (including some new continued fractions) in §7, as well as the new Beta function representation EQ 63, and the four partial fraction expansions (some not new, but all come with new error estimates) in §12-13. The better and newer results tend to be located nearer the end of the paper; many claims stated early on will not be new and are merely preparatory. I will not argue here about what the "best" numerical method is for evaluating the Gamma function (nor what "best" even means), but in at least some senses the partial fraction expansions seem the best ways I've seen.

2 The ratio and product functions

Because of the reflection formula (HOMF 6.1.17)

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z\sin(\pi z)} \tag{1}$$

the product $P(z) = \Gamma(z)\Gamma(-z)$ is easy to compute; therefore to compute $\Gamma(z)$ it is only necessary to find the *ratio* $R(z) = \Gamma(z)/\Gamma(-z)$, and that only in the right half plane $\mathrm{Re}(z) \geq 0$. Indeed it suffices merely to be able to do it in a strip $n/2 \leq \mathrm{Re}(z) \leq (n+1)/2$ where n is any integer, in view also of the incrementing formula $\Gamma(z) = z\Gamma(z-1)$.

Some would argue that the study of $R(z)$ is really just the study of $\Gamma(z)^2$ because

$$R(z) = \frac{-z}{\pi}\sin(\pi z)\Gamma(z)^2. \tag{2}$$

Others would counter that the product and ratio both seem "more natural" than the Gamma function because of their higher symmetry: $P(-z) = P(z)$, $R(-z)R(z) = 1$. (All three of P, R and Γ obey $F(\overline{z}) = \overline{F(z)}$.) So the purpose of this paper is to study $R(z)$, $\Gamma(z)$, $\Gamma(z)^2$, $\ln\Gamma(z)$, and Binet's function $\mu(z)$.

*Non-electronic mail to: 21 Shore Oaks Drive, Stony Brook NY 11790.

[1]My criterion for "newness" is that it is not in HOMF, Gradshteyn and Rhyzhik's *Tables*, the monographs by Erdelyi et al [12], Nielsen [24], Artin [2], Whittaker and Watson [42], the Wolfram Research formula database, and various papers and special function books I've read.

3 Characterizations

Theorem 1 (Previous Characterizations of the Gamma function). *$\Gamma(z)$ is the unique function obeying $\Gamma(1) = 1$, $z\Gamma(z) = \Gamma(z+1)$, and any one of the following:*

1. *$1/\Gamma(z)$ is analytic in the right half plane $\mathrm{Re}(z) > 0$ and bounded by $1/|\Gamma(z)| = O(e^{\pi|z|})$. (Consequence of Carlson's theorem, see footnote 16.)*
2. *$\Gamma(z)$ is analytic in the right half plane and bounded on the strip $1 \le \mathrm{Re}(z) < 2$. (Wielandt [30]).*
3. *$\ln\Gamma(z)$ is concave-$\cup$ for all positive real z. (Bohr and Mollerup 1922, discussed in [2] and [32] p.193).*
4. *$\Gamma(z)$ is absolutely continuous for $1/2 \le z \le 1+\epsilon$ for some $\epsilon > 0$ and obeys Legendre's duplication formula $\Gamma(2z) = \Gamma(z)\Gamma(z+1/2)2^{2z-1}/\sqrt{\pi}$ for real $z > 0$. (Artin [2] as strengthened by Kairies [19]; Kairies indeed does not even require $z\Gamma(z) = \Gamma(z+1)$ provided $\lim_{x\to 0} x\Gamma(x) = 1$ is known.)*

4 Stirling's formula and the Hermite-Sonin-Nörlund shifting trick

This section will give some forms of Stirling's formula having advantages over the usual form either in ease of use, strength, naturalness, or simplicity.

One standard form (HOMF 6.1.40) is

$$\ln\Gamma(z) \;=\; (z-\frac{1}{2})\ln z - z + \ln\sqrt{2\pi} + \mu(z), \qquad \mu(z) \;=\; R_n(z) + \sum_{m=1}^{n} \frac{B_{2m}}{(2m-1)\,2m\,z^{2m-1}} \tag{3}$$

where $\mu(z)$ is *Binet's function* and B_n are the Bernoulli numbers, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, etc. Unfortunately, the series (if extended to $n = \infty$) always diverges, necessitating the use of a finite series truncation and a remainder term $R_n(z)$. Fortunately this remainder term can be expressed in closed form as an integral ([42], [26] p.294, [29], [34], [11] p.46-48; here $B_{2n}(x)$ is a Bernoulli polynomial [HOMF 23])

$$R_n(z) = \frac{(-1)^n}{\pi z^{2n-1}} \int_0^\infty \frac{t^{2n}}{t^2+z^2} \ln\frac{1}{1-e^{-2\pi t}} \mathrm{d}t = \frac{-1}{2n}\int_0^\infty \frac{B_{2n}(x-\lfloor x\rfloor)}{(z+x)^{2n}}\mathrm{d}x = \frac{1}{2n}\int_0^\infty \frac{B_{2n} - B_{2n}(x-\lfloor x\rfloor)}{(z+x)^{2n}}\mathrm{d}x \tag{4}$$

which can be bounded. (And note that $\mu(z) = R_0(z)$.) Known bounds due to Lindelöf (see p.208 of [24]), Stieltjes, and Spira [39], as well as some derived in [42] p.253 and [26] p.294, include:

$$|R_n(z)| \le \frac{|B_{2n+2}|}{(2n+2)(2n+1)|z|^{2n+1}} \begin{cases} 1 & \text{if } 0 \le |\theta| \le \pi/4 \text{ or } 3\pi/4 \le |\theta| < \pi \\ \sec(\theta/2)^{2n+2} & \text{if } |\theta| < \pi \\ \csc(2\theta) & \text{if } |\theta| < \pi \end{cases} \tag{5}$$

$$|R_n(z)| \le \frac{2\,|B_{2n}|}{2n-1} \begin{cases} |\mathrm{Im}(z)|^{1-2n} & \text{if } \mathrm{Re}(z) < 0 \text{ and } \mathrm{Im}(z) \ne 0 \\ |z|^{1-2n} & \text{if } \mathrm{Re}(z) \ge 0 \end{cases} \tag{6}$$

where $z = |z|e^{i\theta}$. Also, if $z > 0$ is real then $R_n(z)$ is less than, but has the same sign as, the first neglected term in the series ([42] p.253; rediscovered about 100 years later by C.Impens [18]).

Thus, if $z > 0$ is real, Stirling's series is valid in the senses that

1. the true value of the left hand side lies strictly *between* any two consecutive series truncations;
2. it is valid in the sense of asymptotic series as $|z| \to \infty$ along any ray with $|\arg z| < \pi$ emanating from the origin in the complex z plane (that is, any truncation of the series yields an approximation whose additive error is asymptotic to the next term); and finally
3. it is valid as $|z| \to \infty$ along any power-law curve (such as, if $p = 2$, a parabola) $x = A - By^p$ where A and $B > 0$ and $p \ge 1$ are fixed reals and $z = x + iy$, in the weaker sense that one can make the error go to zero faster than any desired power $|z|^{-q}$ of $|z|$ by using enough terms (the required number depends on p and q) in the Stirling series.

"Raabe's exact formula" ([25] p.115, [24]) is

$$\int_z^{z+1} \ln\Gamma(x)\mathrm{d}x = z\ln z - z + \ln\sqrt{2\pi}, \qquad |\arg z| < \pi. \tag{7}$$

[42] ex.21 of ch.12 suggests deriving this by showing the z-derivative of both sides is $\ln z$.

Nörlund ([25] p.111) gave the following more general form of Stirling's formula which permits an arbitrary "shift" h with $0 \le h \le 1$:

$$\ln\Gamma(z+h) = \ln\sqrt{2\pi} + (z+h-\frac{1}{2})\ln z - z - \sum_{j=1}^{m-1} \frac{(-1)^j B_{j+1}(h)}{(j+1)j\,z^j} - \frac{1}{m}\int_0^\infty \frac{B_m(t-h-\lfloor t-h\rfloor)}{(z+t)^m}\mathrm{d}t \tag{8}$$

for $|\arg z| < \pi$. Again, $B_n(x)$ are the Bernoulli polynomials.

If $h = 0$ or $h = 1$ this just reduces to the usual Stirling formula because $B_j(0) = (-1)^j B_j(1) = B_j$, and note that $(-1)^n = -1$ if n is odd. The usual Stirling form has the advantage that only odd-j terms occur in the sum because $B_j = 0$ if j is odd and $j \geq 3$.

But the choice $h = 1/2$, due to N.Ya.Sonin and Ch.Hermite [16][38] in 1882-1896, seems even more superior, since it also yields an all-odd series because of (HOMF 23.1.21)

$$B_j(\frac{1}{2}) = (2^{1-j} - 1)B_j, \qquad j = 0, 1, 2, 3, \ldots \tag{9}$$

while *halving* the |coefficient| of the dominant z^{-1} term in the series and indeed decreasing the magnitude of every term (although the effect on terms with large j is tiny).

In view of this it seems worth trying to duplicate the usual analysis of the Stirling-Binet series, but instead for the Hermite-Sonin "more natural" series. That can be done by defining the following "Hermite-Sonin-Nörlund function" to play the role of Binet's function $\mu(z)$:

$$\mathrm{HSN}(z) \stackrel{\text{def}}{=} \ln\Gamma\left(z + \frac{1}{2}\right) - \ln\sqrt{2\pi} - z\ln z + z \;=\; \mu(z + \frac{1}{2}) + z\ln(1 + \frac{1}{2z}) - \frac{1}{2} \;\sim\; \frac{-1}{24z} + \frac{7}{2880z^3} - \frac{31}{40320z^5} + \cdots \tag{10}$$

and then one can get an integral representation for $\mathrm{HSN}(z)$ from any integral representation of Binet's function $\mu(z)$. Usually the parallel expressions for $\mathrm{HSN}(z)$ are slightly simpler than those for $\mu(z)$ and are usually best found by parallel derivations. (For the original derivations, see chapter 12 of [42].)

Binet's function $\mu(z)$ and $\mathrm{HSN}(z)$ obey the following difference equations:

$$\mu(z) - \mu(z+1) = z + \frac{1}{2}\ln\left(1 + \frac{1}{z}\right) - 1, \qquad \mathrm{HSN}(z) - \mathrm{HSN}(z+1) = z\ln\left(1 + \frac{1}{z}\right) + \ln\left(\frac{2z+2}{2z+1}\right) - 1. \tag{11}$$

Corresponding to Binet's first integral (EQ 25) I find

$$\mathrm{HSN}(z) = \int_0^\infty \left(\frac{\exp(x/2)}{e^x - 1} - \frac{1}{x}\right)\frac{e^{-xz}}{x}\mathrm{d}x, \tag{12}$$

and corresponding to Binet's second integral (EQ 26) I find

$$\mathrm{HSN}(z) = -2\int_0^\infty \frac{\arctan(v/z)}{e^{2\pi v} + 1}\mathrm{d}v = -2z\int_0^\infty \frac{\arctan u}{e^{2\pi u z} + 1}\mathrm{d}u, \qquad \mathrm{Re}(z) > 0, \tag{13}$$

and finally for the remainder after n terms of the Hermite-Sonin $h = 1/2$ series (where $\mathrm{HSN}(z) = \tilde{R}_0(z)$ is yet another integral representation of $\mathrm{HSN}(z)$) I find the considerably better-behaved

$$\tilde{R}_n(z) = \frac{(-1)^n}{\pi z^{2n-1}}\int_0^\infty \frac{t^{2n}}{t^2 + z^2}\ln\frac{1}{1 + e^{-2\pi t}}\mathrm{d}t \tag{14}$$

Sonin [38] showed that the remainder from truncating his series was always of the same order or smaller than the corresponding remainder in the Stirling-Binet series, so it should be preferred.

The top two disadvantages of both Stirling's formula and the variant Hermite-Sonin formula are

1. the fact that the series diverges everywhere (exactly how drastically is revealed by the Bernoulli-number asymptotic EQ 30) and
2. Bernoulli numbers are inconvenient.

For a *convergent* continued fraction *not* involving Bernoulli numbers, see EQ 91. See also EQ 55 and the convergent Gudermann series EQs 69-70.

Two convergent forms of Stirling's formula have been proposed but they are not very satisfactory. Binet found (according to [42] p.253 which gives no proof[2])

$$\mu(z) = \frac{c_1}{z+1} + \frac{c_2}{(z+1)^{\hat{2}}} + \frac{c_3}{(z+1)^{\hat{3}}} + \ldots, \qquad c_n = \frac{1}{n}\int_0^1 x^{\hat{n}}(x - \frac{1}{2})\mathrm{d}x \tag{15}$$

where $(x)^{\hat{n}} = x(x+1)\cdots(x+n-1)$ denotes the rising factorial, in a better notation than the horrible but unfortunately more commonly employed "Pochhammer symbol $(x)_n$." This converges if $\mathrm{Re}(z) > 0$ and is equivalent to

$$\mu(z) = \int_0^1 {}_3F_2\left(\begin{matrix}1, 1, x+1 \\ 2, z+2\end{matrix}\;;1\right)\frac{(x - 1/2)x}{z+1}\mathrm{d}x. \tag{16}$$

[2] It is possible to prove it via EQ 16 after re-expressing the ${}_3F_2$ in terms of polygamma functions.

Unfortunately the coefficients c_n are mysterious and the convergence is very slow – power law rather than exponential – because the nth summand behaves roughly like n^{-z-1} (as one can see by considering the ratio of the two Pochhammer symbols).

T.J.Stieltjes suggested the continued fraction (HOMF 6.1.48)

$$\mu(z) = \frac{a_0}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\,\frac{a_3}{z+\cdots}\,, \qquad a_0 = \frac{1}{12},\ \ a_1 = \frac{1}{30},\ \ a_2 = \frac{53}{210},\ \ a_3 = \frac{195}{371},\ \ a_4 = \frac{22999}{22737} \tag{17}$$

which he proved converges if $\mathrm{Re}(z) > 0$ (convergence follows from "Carleman's criterion," see theorem 4.58 and §9.6 of [13]) but the coefficients a_n here are quite mysterious, certainly much more mysterious than the Bernoulli numbers from which they may be deduced. Char [9] tabulated the first twelve a_n in exact form and gave high-precision decimal approximations for the first 41, as well as tabulating b_j and c_j for $j = 0, 1, \ldots, 20$ where

$$\mu(z) = \frac{z}{12z^2 + b_0-}\,\frac{c_1}{12z^2 + b_1-}\,\frac{c_2}{12z^2 + b_2 - \cdots}\,, \qquad b_j = 12(a_{2j+1} + a_{2j}), \quad c_j = 144 a_{2j} a_{2j-1}. \tag{18}$$

Cizek and Vrscay, in a 2-page note [10] containing claims without any proofs[3] observed that

$$\frac{1}{3}\int_0^\infty \frac{e^{-4zt}}{\cosh t}\mathrm{d}t = \frac{b_0}{z+}\,\frac{b_1}{z+}\,\frac{b_2}{z+}\,\frac{b_3}{z+\cdots}\,, \qquad b_0 = \frac{1}{12},\ \ b_n = \frac{n^2}{16}\ \text{ for } \ n \ge 1 \tag{19}$$

had been shown by Stieltjes if $\mathrm{Re}(z) > 0$ and then claimed that $|a_n - b_n| = O(n)$, that all the a_n and b_n are positive rationals, and that $a_n - b_n$ alternates in sign for $n \ge 1$.

The Cizek-Vrscay claim that all $a_n > 0$ and $b_n > 0$ is true – this may be proved from Stieltjes's theory of continued fractions arising from integrals and his associated "$(0, \infty)$ moment problem," see §87-88 of [41] and §9.6 of [13]. (Also, the claim that all a_n and b_n are rational is trivially true.) Hence the convergents of EQ 17 successively provide upper and lower bounds on $\mu(z)$ if $z > 0$ is real, with the upper bounds monotonically decreasing and the lower bounds monotonically increasing as you take more decks in the continued fraction. (Sadly, a large number of papers in the subsequent 100+ years have consisted of proving $\Gamma(z)$-bracketing bounds a lot weaker and more random-looking than Stieltjes already had.)

It took a while, but I was finally able to prove the remaining Cizek-Vrscay claim[4] that $a_n \sim n^2/16$ by appealing to the deep and recently established [22] "Freud conjectures" from approximation theory.

continued fractions). The nth coefficient a_n in EQ 17 (and the nth coeffcient g_n in EQ 23) is asymptotic for large n to $n^2/16$. Consequently, the additive error in the approximation of $\Gamma(z)$, for fixed z with $\mathrm{Re}(z) > 0$, got by using n decks of either continued fraction, is asymptotically proportional to $n^{-4\mathrm{Re}(z)}$ when $n \to \infty$.

Remarks. Before commencing the proof, we list some incomplete evidence for the theorem's validity, which motivated trying to prove it. First, Char's [9] numerical values for $a_0, \ldots, a_{40}$ suggest $|a_n - n^2/16| \approx 0.003n$, e.g. $a_{10} \approx 6.28$, $a_{20} \approx 25.06$, $a_{40} \approx 100.12$. Second, Henrici & Pfluger [15]'s EQ 3.5 shows that $a_n \lesssim (n/\pi)^2$ is true at least in the geometric-mean sense that $a_1 a_2 \cdots a_n \lesssim n!^2/\pi^{2n}$; this is an upper bound of the same nature as, although weaker than, the Cizek-Vrcsay claim. Third, we can prove $a_n \sim \kappa n^{2-\epsilon}$ impossible for $\kappa, \epsilon \in (0, 1)$ because indeed we can prove $\prod_{k=1}^n (a_{k-1} + a_k) \gtrsim (2n-2)!/(4\pi^2)^n$ by simply considering the Wallis recurrence relations for computing the rational function corresponding to a n-term continued fraction. Each deck can increase the magnitude of the coefficients of the rational function by at most a factor of $a_n + a_{n-1}$, and then the nth term in the asymptotic series can be estimated by choosing z to be a value causing that term to dominate the preceding ones. This shows that if the z_n are increasing, then $a_n \gtrsim n^2/(2\pi^2)$, which is a matching lower bound weaker than, but of the same character as, the Cizek-Vrscay claim.

Proof sketch. The idea (§87-88 of [41] and §9.6 of [13]) is that the a_n correspond to the coefficients in the 3-term recurrence relation obeyed by the monic polynomials orthogonal with respect to the "Stieltjes measure" in $(0, \infty)$. Now the Stieltjes measure in this case is immediately seen to be asymptotically identical to an appropriately chosen and scaled "Laguerre measure" $x^\alpha \exp(-x)\mathrm{d}x$ when x is large. (Actually it is best for purposes below to use a square-root change of variables to convert it to a "Hamburger moment problem" on $(-\infty, \infty)$ instead, and instead with asymptopia to a "Freud measure." That is permissible because of the constant-parity nature of the Stirling asymptotic power series, or, in moment language, the fact that only moments of *one* parity are nonzero.)

Now suppose – or hope – that what happens when $|x|$ is *small* is asymptotically irrelevant to all integrals. That forces the recurrence coefficients to be asymptotically the same as the ones that are already known in closed form for the "Laguerre polynomials" [HOMF 22]. (I am intentionally not providing details because they *are not needed*; exactly what I am saying is all you need to know and delving further into the details is a waste of time and liable to lead to errors.)

This would immediately prove it. The only problem is – how do we justify this "hope"? I tried unsuccessfully for a long time to do that. It seemed difficult. But then it dawned on me that this indeed *was* a difficult – but *already solved* – problem: what we need is merely a special case of the long-awaited proof of the "Freud conjectures" [22]!

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[3] Later publications were promised but never provided, and according to the *Science Citation Index*, [10] was never cited during the succeeding 23 years.

[4] According to Cizek and Vrcsay supposedly their claims can be shown by analysing processes for converting the two asymptotic series to these two continued fractions, keeping track of comparative inequalities as we go. It will be nice if it really can be done that way, but our approach is entirely different.

The First Freud Conjecture is that if a positive weight function $W(x)$ on the real line obeys

$$\ln W(x) \sim -(x^2)^{-m/2}, \qquad |x| \to \infty, \qquad \text{for fixed real } m > 0 \tag{20}$$

then the coefficients a_n in the recurrence relation

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x) \tag{21}$$

obeyed by the orthonormal polynomials with respect to $W(x)$ are asymptotic for large n to $n^{1/m}$ times a certain explicitly stated function of m only:

$$a_n \sim n^{1/m}\left(\frac{\Gamma(m/2)\Gamma(m/2+1)}{\Gamma(m+1)}\right)^{1/m}. \tag{22}$$

This conjecture was finally proven [22] in 1988.[5]

It now follows fairly easily once you know that $a_n \sim n^2/16$, that the ultimate convergence rates of both continued fractions are of power-law style, with error after n decks ultimately declining proportionally to $n^{-4\mathrm{Re}(z)}$. (This may be shown by an eigen-ratio analysis of the 2×2 matrices that define the Wallis recurrence relations for the continued fraction; some similar analyses of convergence rates for continued fractions are in [4], e.g. see their exercise 8.58b p.416 with reference to their EQ 8.4.1 p.395.) Q.E.D.

Henrici & Pfluger [15] proved quantitative versions of Carleman's criterion[6] and used them to show that the |error| after n decks of Stieltjes' CF decreases at least as fast as $1/\sqrt{\ln n}$ after n decks. This is, of course, a much weaker result than what we just proved.

As usual, it seems superior to CFify the *Hermite-Sonin*, rather than Binet, function. When we do, we get the new result

$$\mathrm{HSN}(z) = \frac{g_0}{z+}\,\frac{g_1}{z+}\,\frac{g_2}{z+}\,\frac{g_3}{z+\cdots}, \qquad g_0 = \frac{-1}{24},\ g_1 = \frac{7}{120},\ g_2 = \frac{1517}{5880},\ g_3 = \frac{164715}{297332} \tag{23}$$

$$g_4, g_5, g_6, g_7 = \frac{2221550065}{2198879364}, \frac{3711235756721941}{2392088434778328}, \frac{260989522174000033487601}{115352318324821953965 20}, \frac{430585991407918092965025264911309}{141209860872983253300302530483230}. \tag{24}$$

The Stieltjes moment theory as in §9.6 of [13] (but this time starting from $\mathrm{HSN}(z) = \tilde{R}_0(z)$ in our EQ 14) again shows that $g_n > 0$ for all $n \ge 1$. Again Carleman's criterion again shows convergence for all complex z with $\mathrm{Re}(z) > 0$ and the Henrici-Pfluger quantitative version of Carleman's criterion again shows the |error| decreases at least proportionally to $1/\sqrt{\ln n}$ after n decks. Thus again the CF convergents provide rigorous upper and lower bounds (which keep contracting toward each other the more decks one takes) if z is real and positive. We again note that the g_n are asymptotic to $(n/4)^2$ when $n \to +\infty$ (and conjecturally[7] monotonically increasing) again forcing the ultimate behavior of the nth |error| to be power law, asymptotically proportional to $n^{-4\mathrm{Re}(z)}$ when $n \to \infty$.

We now are going to show how to get convergent forms of both Stirling's formula and its Sonin $h = 1/2$ variant, both with *no* mysterious coefficients, but their convergence rates again will be disappointing.

Before we do that let us quickly review how Stirling's formula was derived. There are many known ways to obtain it. Here is one that is slightly more direct than usual. Binet's function $\mu(z)$ is defined in two ways by Binet's integral representations ([33], [42] p.249-251)

$$\mu(z) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right)\frac{e^{-xz}}{x}\mathrm{d}x, \tag{25}$$

$$\mu(z) = 2\int_0^\infty \frac{\arctan(v/z)}{e^{2\pi v} - 1}\mathrm{d}v = 2z\int_0^\infty \frac{\arctan u}{e^{2\pi uz} - 1}\mathrm{d}u, \qquad \mathrm{Re}(z) > 0. \tag{26}$$

In Binet's second integral, employ the geometric and arctan-Maclaurin series

$$\frac{1}{e^y - 1} = e^{-y} + e^{-2y} + e^{-3y} + \cdots, \qquad \mathrm{Re}(y) > 0 \tag{27}$$

$$\arctan(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots, \qquad |u| < 1 \tag{28}$$

[5] The second Freud conjecture gives asymptotic bounds on the zeros of these polynomials and was also proved, by E.A.Rahmanov in Math USSR Sb. 47 (1984) 155-193.

[6] We should note that Carleman's CF-convergence critierion, although satisfied, is just *barely* satisfied for this CF.

[7] Incidentally, note that the derivative of $\Psi(x) = (\mathrm{d}/\mathrm{d}x)\ln\Gamma(x)$ is (after a Nörlundian half-shift)

$$\Psi'(x + \frac{1}{2}) = \frac{1}{x}\,\frac{t_1}{x+}\,\frac{t_2}{x+}\,\frac{t_3}{x+\cdots} \sim \frac{1}{x} + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(1/2)}{x^{j+2}}, \qquad t_n = \frac{n^4}{4(2n-1)(2n+1)} \sim (\frac{n}{4})^2$$

was shown independently by Ramanujan, Rogers, Nörlund, and Stieltjes between 1885 and 1920. Note that the *same* large-n asymptotics of the nth CF coefficient occurs for *both* $\ln\Gamma(x + 1/2)$ and its second derivative $\Psi'(x + 1/2)$, and both also have the same jth coefficients in their asymptotic series, aside from a factor of $(j+1)j$. The reason these two CFs have asymptotically identical coefficients is that both their Stieltjes measures are asymptotic to same-scaled Laguerre measure.

Multiply the two series and integrate term-by-term with the aid of Euler's integral

$$\int_0^\infty x^n e^{-kx} \mathrm{d}x = \frac{n!}{k^{1+n}}, \quad n > -1, \, \mathrm{Re}(k) > 0 \tag{29}$$

and then re-sum the formerly-geometric series with the aid of Euler's zeta(even) formula (HOMF 23.2.16)

$$B_{2n} = \frac{2\,(2n)!}{\pi^{2n} 4^n} \left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \cdots\right] \tag{30}$$

to get Stirling's series. (Essentially the same method may be used to derive the Soninian 1/2-shifted variant series from EQ 13; one uses the alternating Riemann zeta function instead of the usual one, see HOMF 23.2.19.) This method has the advantage that it gets all terms of Stirling's series at once, but it must be admitted that this is purely a formal power series manipulation and not rigorously justified. That is because only the interval $[0, z]$ is within the domain of convergence of the arctan-Maclaurin series; for the remaining part $[z, \infty]$ of the line of integration in EQ 26, we are resting on air. (And that is also, of course, one of the underlying reasons why Stirling's series diverges everywhere.) That is, however, ok if we are merely interested in asymptotic behavior as $z \to \infty$, essentially because the bad second part of the integral makes a negligible contribution in that limit. More precisely: by repeatedly integrating Binet's integral (or more directly the R_0 integral) by parts, we get the first exact expression EQ 4 for the remainder term.

One could also have based everything on Binet's first integral by using

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}, \quad |x| < 2\pi, \tag{31}$$

or more generally see HOMF 23.1.1 for a form that permits also handling EQ 12.

In both these derivations, one can obtain rigorous error bounds in other ways. One is to employ Taylor's series formula *with remainder*

$$f(x) = R_N(x) + \sum_{n=0}^{N} f^{(n)}(0)\frac{x^n}{n!}, \qquad R_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N \mathrm{d}t = f^{(N+1)}(b)\frac{b^N}{(N+1)!}, \quad 0 < b < x. \tag{32}$$

In that case the fact that part of the line of integration lies outside the domain of series convergence does not matter, since we only deal with truncated series plus a Taylor-remainder bound.

The main reason we are saying all this is to point out that if, in the derivation based on Binet's second integral EQ 26 (or its Soninian variant EQ 13), we instead employ *this* arctan series (HOMF 4.4.42, which converges for all u with $|u^2| < |1 + u^2|$, i.e. in a much larger region than our former series for arctan, which, importantly for us, contains the entire real axis)

$$\arctan u = \frac{u}{1+u^2}\left[1 + \frac{2}{3}(\frac{u^2}{1+u^2}) + \frac{2\cdot 4}{3\cdot 5}(\frac{u^2}{1+u^2})^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}(\frac{u^2}{1+u^2})^3 + \cdots\right] \tag{33}$$

then we can get a new convergent version of Stirling's formula. (Note that both of the "input" series EQ 27 and 33 converge exponentially everywhere in the interior of the interval of integration.) We work in the same way as before except that, unfortunately, the term-by-term integration now needs to be done with

$$\int_0^\infty \frac{x}{1+x^2}\left(\frac{x^2}{1+x^2}\right)^n e^{-cx}\,\mathrm{d}x \;=\; \frac{1}{2\sqrt{\pi}\,n!}\,G^{3,1}_{1,3}\left(\frac{c^2}{4}\,\middle|\,\begin{matrix}-n\\0,0,1/2\end{matrix}\right) \tag{34}$$

where G is Meijer's G-function, described in [27] vol.3 p.793. Although one quails in horror at the prospect of a series (even a convergent one) whose terms are Meijer G-functions, these particular Meijer G-functions are much nicer than fully general ones. They may be evaluated by means of a recurrence-scheme by Acton [1]. Acton defines the integrals

$$F_n(c) = \int_0^\infty \frac{x}{1+x^2}\left(\frac{x^2}{1+x^2}\right)^n e^{-cx}\,\mathrm{d}x, \quad G_n(c) = \int_0^\infty \frac{1}{1+x^2}\left(\frac{x^2}{1+x^2}\right)^n e^{-cx}\,\mathrm{d}x, \quad H_n(c) = \int_0^\infty \left(\frac{x^2}{1+x^2}\right)^n e^{-cx}\,\mathrm{d}x \tag{35}$$

where $F_n(c)$ is the integral in EQ 34 that we want, and by integration by parts with frequent use of the relations

$$\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^2}{1+t^2}\right] = \frac{2t}{(1+t^2)^2}, \tag{36}$$

Acton finds these recurrences

$$F_{n-1} = F_n + \frac{c}{2n}H_n, \quad G_{n-1} = \frac{2nG_n + cF_{n-1}}{2n-1}, \quad H_{n-1} = H_n + G_{n-1}. \tag{37}$$

Acton then starts at some suitably large N with $F_N = 1$, $G_N = H_N = 0$ and uses these recurrences backwards to compute F_n, G_n, H_n for $n = N-1, N-2, \ldots, 1, 0$. He then argues that in the limit $N \to \infty$ the exact answers will be rapidly approached at any fixed n, *except* that all three of F_n, G_n, H_n will be erroneous by the same common factor (which evidently is independent of n). This common factor can be removed by using the exact result $H_0(c) = 1/c$ to normalize all answers. Note that in this fashion we compute *all* of the $F_n(c)$ at the same cost as just one of them, in $O(N)$ arithmetic operations. Convergence is faster if $\mathrm{Re}(c)$ is large.[8] (Acton: "As a rough guide, $N = 150/c$ will yield approximately 10 significant figures in F_0 and G_0.") We then have

Theorem 3 (New convergent version of Stirling's formula).

$$\mu(z) = 2z \sum_{k=1}^{\infty} \left[F_0(2\pi zk) + \frac{2}{3} F_1(2\pi zk) + \frac{2\cdot 4}{3\cdot 5} F_2(2\pi zk) + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} F_3(2\pi zk) + \cdots \right]. \tag{38}$$

where $F_n(x)$ is defined above. The Soninian variant form is

$$\mathrm{HSN}(z) = 2z \sum_{k=1}^{\infty} (-1)^k \left[F_0(2\pi zk) + \frac{2}{3} F_1(2\pi zk) + \frac{2\cdot 4}{3\cdot 5} F_2(2\pi zk) + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} F_3(2\pi zk) + \cdots \right] \tag{39}$$

Proof. We have already shown how to obtain this and the formal manipulations are justified by the absolute convergence of all series throughout the domain of integration.

To prove convergence, and to get an idea of its rate of convergence, note that as $n \to +\infty$ with c bounded below by a positive constant, $F_n(c) \to 0+$ nearly exponentially, that is $\log F$ behaves at least proportionally to $-n^{1/3}$. (Proven by a local analysis around the maximum of its integrand, which is located at $x \sim (n/c)^{1/3} 54^{1/3}/3$.) That proves that each inner sum converges rapidly. (And all the summands in each inner sum may be computed simultaneously in *one* pass of Acton's recurrence scheme.)

On the other hand, as $c \to +\infty$ with n fixed, $F_n(c) \sim c^{-2-2n}\Gamma(2+2n) = \int_0^\infty x^{1+2n}\exp(-cx)\mathrm{d}x$. This proves the outer sum in EQ 38 converges everywhere in the halfplane $\mathrm{Re}(z) > 0$ like $\sum_{k\ge 1}(zk)^{-2}$, which is a rather disappointing rate. The outer sum in EQ 39 converges everywhere in the halfplane $\mathrm{Re}(z) > 0$ like $\sum_{k\ge 1}(-1)^k(zk)^{-2}$, which also is disappointing, although somewhat better because of the possibility of combining terms in pairs to get a decrease like the -3 power. ❄

As you can see, the fact that both "input" series converge exponentially everywhere in the interior of the interval of integration yielded only a slowly-convergent "output" series, contrary to plausible hopes.[9]

So in conclusion, of the convergent forms of Stirling formula we've mentioned, all are disappointing but probably Stieltjes's continued fraction EQ 17 or its Soninian variant EQ 23 is the best since it converges the fastest, has the enjoyable property that its convergents alternately provide upper and lower bounds if $z > 0$ is real, and the formula is the simplest – *provided* the magic coefficients a_n are known. (In principle the first n of them may be computed from EQ 31 by means of fast power series and CF algorithms in $O(n(\log n)^2)$ arithmetic operations.)

Here are four *easy-to-remember* Stirling-like approximations. C.W. Gosper once pointed out that

$$z! \;=\; z\Gamma(z) \;=\; \sqrt{(2z+1/3)\pi}\,(\frac{z}{e})^z\,\left[1 + O(|z|^{-2})\right]. \tag{40}$$

Following the Sonin "half-shift" idea, we find

$$\Gamma(z+\frac{1}{2}) = \sqrt{2\pi}\left(z^2 - \frac{1}{12} + \frac{1}{120z^2} - \frac{37}{18144z^4} + \frac{521}{388800z^6} - \frac{107353}{59875200z^8} + \cdots\right)^{z/2} e^{-z}\left[1 + O(|z|^{1-2n})\right]. \tag{41}$$

where n terms of the series are employed. (Just using the first two terms to get an error term of order z^{-3} was once suggested by Wolfgang Schuster.) Robert H. Windschitl, in a 2002 web post, pointed out that

$$z! \;=\; z\Gamma(z) \;=\; \sqrt{2\pi z}\,(\frac{z}{e})^z (z\sinh\frac{1}{z})^{z/2}\left[1 + O(|z|^{-5})\right]. \tag{42}$$

Our effort to out-cute these is:

$$\Gamma(z+\frac{1}{2}) \;=\; \sqrt{2\pi}\,(\frac{z}{e})^z (2z\tanh\frac{1}{2z})^{z/2}\left[1 + O(|z|^{-5})\right]. \tag{43}$$

The agreement between Windschitl's simple expression and the official exponentiated Stirling power series – they have the same coefficients of z^{-k} for $k = 0, 1, 2, 3, 4$ – is either a surprising coincidence, *or* a sign of some unknown underlying structure. (Ditto re our EQ 43.) Which?

[8] It would also be possible in principle to use Acton's recurrences *forwards* starting with exact closed forms for F_0 and G_0 (and $H_0 = 1/c$) in terms of "exponential, sine, and cosine integrals."

[9] The problem is that if several series converge at different rates, and all are computed, then the slow one is the one that dominates the computer's workload. Our integral indeed combines series converging at different rates and the slowest behavior dominates the picture.

z	Windschitl	Cute	$z!$
0.0	undefined or 0*	1.00429	1
0.5	0.88214	0.88652	0.88623
1.0	0.999658	1.000057	1
1.5	1.329262	1.329361	1.329340
2.0	1.9999683	2.0000107	2

Figure 4.1. Comparison of two concise approximate Stirling-like formulae for $z!$ by Windshitl (EQ 42) and us (EQ 43) with the truth. Although these formulae are designed for asymptotic validity when $|z| \to \infty$ with $|\arg z| < \pi$, they (especially our's) perform remarkably well even for small real z. *: Although Windshitl's formula for $z!$ is undefined if $z \le 0$, it gives 0 in the *directional limit* $z \to 0+$, as opposed to the correct $0! = 1$. Similarly our "cute" formula is undefined for $(-1/2)! = \sqrt{\pi} \approx 1.8$ but gives $\sqrt{2\pi} \approx 2.5$ in the directional limit. ▲

Here is a *convenient* enhanced version of HOMF 6.1.39:

$$\Gamma(az+b) \;=\; \sqrt{2\pi} e^{-az} (az)^{az+b-1/2} \left(1 + \frac{6b^2-6b+1}{12a\, z} + \frac{36b^4 - 120b^3 + 120b^2 - 36b + 1}{288 a^2\, z^2} + O(|z|^{-3})\right), \tag{44}$$

Although these particular results are new, the techniques for proving them are old; a large number of ways to prove results of this ilk are explained in numerous textbooks [2][24][42],including Euler-Maclaurin summation and repeated integration by parts of Binet's integral representation. One can also use the saddlepoint method [4] for Hankel's integral EQ 79 or for Euler's integral in combination (if necessary) with the reflection formula, or just parasite off the known truth of plainer versions of Stirling's formula.

5 $R(z)$'s behavior in the complex plane

$R(z)$ is meromorphic throughout the complex plane and would be entire analytic except that it has a pole at each negative integer $z = -k$, with residue $(-1)^k k/k!^2$. We have $R(z) = 0$ at each positive integer $z = k$, and $R(z)$ assumes rational-number values at half integers e.g. $R(1/2) = -1/2$,

$$R(k + \frac{1}{2}) \;=\; 8(\frac{-1}{16})^{k+1}(2k+1)!\binom{2k}{k}, \quad k = 1, 2, 3, \ldots \tag{45}$$

and $R(k + 1/2)$ for $k = -1, -2, \ldots$ can be handled via $R(z)R(-z) = 1$. (EQ 45 can be derived from the Gamma function reflection and duplication formulas.)

Note that $R(0) = -1$ (and not $+1$ as one might naively think) since we hereby define $R(0)$ to be the *limit* of $R(z)$ as $z \to 0$. $|R(z)| = 1$ all along the imaginary axis $\mathrm{Re}(z) = 0$. Along any ray from the origin into the $\mathrm{Re}(z) < 0$ halfplane (other than the negative real axis) $|R(z)|$ declines ultimately exponentially or faster; and hence, along any ray from the origin into the $\mathrm{Re}(z) > 0$ halfplane (other than the positive real axis) $|R(z)|$ increases ultimately exponentially or faster. More precise statements can be got from EQ 44.

6 Euler-Weierstrass product formula

$$R(z) = e^{-2\gamma z} \prod_{n=1}^{\infty} \frac{n-z}{n+z} e^{2z/n} \tag{46}$$

where $\gamma = -\Gamma'(1) = 0.5772156649\ldots$ is the Euler-Mascheroni constant. This converges (in the Riemann sphere topology, so that we do not have to make a proviso about "except in the neighborhood of the poles") for all complex z. However, the convergence is slow, reminiscent of the convergence of $\sum_{n\ge1} n^{-3}$. The main reason we are pointing this out is because this is still superior to the convergence of the Euler-Weierstrass product for the Gamma function itself (HOMF 6.1.3), which instead behaves like $\sum_{n\ge1} n^{-2}$.

Another consequence of the Euler-Weierstrass product HOMF 6.1.3 is this product representation of the Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{x+y}{xy} \prod_{n=1}^{\infty} \frac{(n+x+y)n}{(n+x)(n+y)} = \frac{x+y}{xy} \prod_{n\ge1} \left[1 - \frac{xy}{(n+x)(n+y)}\right] \tag{47}$$

which has the advantages that it avoids all mention of exp, ln, and γ, its convergence is immediately obvious, and that if x and y are positive integers, it telescopes down to a finite product.

7 Completely general (and some special) reflection and shift formulas, doubling

Some other versions of the usual Γ reflection formula are

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},\quad -z\Gamma(z)\Gamma(-z)=\Gamma(z)\Gamma(1-z)=\pi\csc(\pi z),\quad \Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z)=\pi\sec(\pi z) \tag{48}$$

For R, in addition to the obvious $R(z)R(-z)=1$, there is also an additional less-obvious reflection formula concerning the mirror-point 1/4. The formula HOMF 15.1.20, which Gauss proved by noticing that when $x=1$ the standard representation HOMF 15.3.1 of ${}_2F_1$ as an integral happens to coincide with Euler's beta-function-defining integral HOMF 6.2.1, is

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\qquad \mathrm{Re}(c-a-b)>0. \tag{49}$$

If we take with $c=z$, $b=2c$, $a=-1/2$ this becomes

$$R(z)R(\frac{1}{2}-z)\;=\;\left(z-\frac{1}{2}\right)\,{}_2F_1(\frac{-1}{2},2z;z;1) \tag{50}$$

which can be viewed as a reflection formula for $R(z)$. Unfortunately the series (HOMF 15.1.1) defining ${}_2F_1(x)$ converges slowly or not at all at $x=1$, in contrast to its geometric-style convergence if $|x|<1$. Specifically, the hypergeometric series in EQ 50 converges absolutely if $\mathrm{Re}(z)<1/2$.

HOMF 15.1.24 arises from a quadratic transformation (HOMF 15.3.22) of a special case of EQ 49 and may be viewed as a "shift by 1/2" formula for the Gamma function:

$$\frac{x\Gamma(x)}{\Gamma(x+1/2)}\;=\;\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\;=\;\frac{1}{\sqrt{\pi}}\,{}_2F_1(1,2x;1+x;\frac{1}{2}). \tag{51}$$

Because of the argument 1/2 in the ${}_2F_1$, we can sigh in relief because this time we have an exponentially convergent hypergeometric series.[10] It therefore also is expressible as a special case of Gauss's continued fraction ([25] p.451, [20]) for ${}_2F_1$ ratios

$$=\;(\frac{1}{\sqrt{\pi}})\,\frac{x}{x-}\;\frac{(2x)(x)/2}{x+1+}\;\frac{1(x-1)/2}{x+2-}\;\frac{(2x+1)(x+1)/2}{x+3+}\;\frac{2(x-2)/2}{x+4-}\;\frac{(2x+2)(x+2)/2}{x+5+}\;\frac{3(x-3)/2}{x+6-}\;\frac{(2x+3)(x+3)/2}{x+7+\cdots} \tag{52}$$

Beautifully, this continued fraction *terminates* (i.e. some partial numerator is 0) with an exact answer if x is any integer or[11] negative half-integer, (albeit $x=0$ must be handled via the limit $x\to 0$ as we have written it, but divide the first and second numerators and first denominator by x to fix that blemish). When it does not terminate, it converges – ultimately in geometric fashion – for each complex x, with the ultimate geometricity well underway after about $4|x|+2$ initial decks.

EQ 51 may also be viewed as giving a continued fraction for central binomial coefficients because

$$\binom{2x}{x}=\frac{4^x}{\sqrt{\pi}}\,\frac{\Gamma(x+1/2)}{\Gamma(x+1)}=4^x\left[1-\frac{x}{x+1+}\;\frac{1(x-1)/2}{x+2-}\;\frac{(2x+1)(x+1)/2}{x+3+}\;\frac{2(x-2)/2}{x+4-}\;\frac{(2x+2)(x+2)/2}{x+5+\cdots}\right] \tag{53}$$

where the continued fractions in EQs 52 and 53 match after the first deck of the latter. There are more ways to represent this quantity as a continued fraction. From HOMF 15.1.21 and Gauss's continued fraction we find

$$\frac{\Gamma(a/2-b)\Gamma(a/2+1/2)}{\Gamma(1/2+a/2-b)\Gamma(a/2+1)}\;=\;\frac{2}{a-b}\,\frac{{}_2F_1(b+1,a+1;a-b+1;-1)}{{}_2F_1(b+1,a;a-b;-1)}\;= \tag{54}$$

$$=\;\frac{2}{a-b+}\;\frac{(b+1)(0-b)}{a-b+1+}\;\frac{(a+1)(a-2b)}{a-b+2+}\;\frac{(b+2)(1-b)}{a-b+3+}\;\frac{(a+2)(a-2b+1)}{a-b+4+}\;\frac{(b+3)(2-b)}{a-b+5+}\;\frac{(a+3)(a-2b+2)}{a-b+6+}\;\frac{(b+4)(3-b)}{a-b+7+\cdots}$$

The special case $a=2x$, $b=-1/2$ yields

$$16^{-x}\pi\binom{2x}{x}^2\;=\;\frac{\Gamma(x+1/2)^2}{\Gamma(x+1)^2}\;=\;\frac{4}{4x+1}\,\frac{{}_2F_1(1/2,2x+1;2x+3/2;-1)}{{}_2F_1(1/2,2x;2x+1/2;-1)}\;= \tag{55}$$

$$=\;\frac{2}{2x+1/2+}\;\frac{(1/2)^2}{2x+3/2+}\;\frac{(2x+1)^2}{2x+5/2+}\;\frac{(3/2)^2}{2x+7/2+}\;\frac{(2x+2)^2}{2x+9/2+}\;\frac{(5/2)^2}{2x+11/2+}\;\frac{(2x+3)^2}{2x+13/2+}\;\frac{(7/2)^2}{2x+15/2+\cdots}$$

[10] It is also possible to get shift-by-1/2 formulae which involve much slower-converging series by using HOMF 15.1.21 or 15.1.20.

[11] When x is a negative half-integer, the CF outputs 0, which it accomplishes by dividing by 0 in internal decks.

This continued fraction terminates after a finite number of decks with the exact answer (namely ∞ or 0, depending on parity) if $2x$ is a negative integer. For the remaining x: although the two series defining the hypergeometric functions do not exponentially converge, the continued fraction does[12] – for every complex x.

These formulas for the central binomial coefficient may also be thought of as argument-*halving* (or *doubling*) formulas for the factorial function.

Also, regarding EQ 55 as a Stirling-like asymptotic formula for $(2x)!/x!^2$ (but which has the advantage that it converges!), we can use it to *derive* Stirling's formula for $x!$ by solving for the coefficients in its power series one by one. This can be thought of as proving a fascinating new identity satisfied by the coefficients in that series.

There are also a large number of "shift by 1/6" formulae involving hypergeometric series with different convergence rates. Here are two of the fastest: Let $L = (2-\sqrt{3})/4 \approx 0.0669872981$.

$$\frac{\Gamma(x+1/6)}{\Gamma(x)} = \frac{\Gamma(3/4)}{\Gamma(7/12)\sqrt{2}\sin(5\pi/12)}(3-24L)^{1/4-x}\,{}_2F_1(\frac{1}{2}-2x,\frac{1}{2}+2x;1-x;L) \tag{56}$$

$$\frac{\Gamma(x+1/6)}{\Gamma(x)} = \frac{3\Gamma(2/3)}{2\sqrt{3\pi}\sqrt[3]{2}}\left(\frac{16}{27}\right)^x {}_2F_1(\frac{1}{3}-2x,\frac{2}{3}-2x;1-x;\frac{-1}{8}). \tag{57}$$

These are theorems 37 and 32 of Ekhad and Zeilberger [43]. They present "WZ-style" concise computer-generated proofs of all their theorems in the terminating-series cases where their variable "n" is an integer, and then validity for noninteger n comes from Carlson's theorem [3].

The complicated constant factors in these expressions are rapidly computable to high precision because they are known to be expressible in terms of algebraic numbers, π, and values of the complete elliptic integral $K(k)$ that may be got via the fast Gauss AGM iteration [7]. I.e. by the (original) reflection formula in the first two cases, and also Gauss's n-tuplication formulas in the third case, we have

$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{3},\quad \Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2},\quad \Gamma(\frac{1}{4})\Gamma(\frac{7}{12}) = \sqrt{\pi\sqrt{2\sqrt{3}}}\sqrt{\sqrt{3}-1}\,\Gamma(\frac{1}{3}),\quad \Gamma(\frac{1}{6}) = 2^{-1/3}\sqrt{3/\pi}\,\Gamma(\frac{1}{3})^2 \tag{58}$$

and

$$\Gamma(\frac{1}{3}) = \frac{\pi^{1/3}2^{7/9}}{3^{1/12}}K\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{1/3},\quad \Gamma(\frac{1}{4}) = 2\pi^{1/4}K\left(\frac{1}{\sqrt{2}}\right)^{1/2},\quad \sin\left(\frac{5\pi}{12}\right) = \frac{(1+\sqrt{3})\sqrt{2}}{4}. \tag{59}$$

see [7][8][21][36]. (Other, slower, shift-by-1/6 formulae arise from HOMF 15.1.29 and 15.1.31 and numerous theorems in [43].)

Because of the shift-by-1/6 formulae, it suffices to be able to compute $\Gamma(z)$ in a 1/6-wide strip, and indeed when we also use one of the reflection formulas we realize that a 1/12-wide strip, e.g. any $n/12$-shift of $0 \le \mathrm{Re}(z) \le 1/12$, suffices.

But all the above are merely special nice cases. We can use HOMF 15.1.24 in a different way to get a fully general reflection formula. The result is an exponentially-convergent series allowing reflecting about *any* complex value M (except if $3/2-2M$ is a positive integer, but we shall see how to handle those cases soon) if $\Gamma(2M-1/2)$ is known:

$$\Gamma(M-x)\Gamma(M+x) \;=\; \sqrt{\pi}\,\Gamma\left(2M-\frac{1}{2}\right)/{}_2F_1(2M-1-2x,2M-1+2x;2M-\frac{1}{2};\frac{1}{2}), \tag{60}$$

and we can shift by *any* complex value S (except if $1/2-S$ is a positive integer, but then use $-S$ or employ EQ 51-55 above) if $\Gamma(S+1/2)$ is known:

$$\frac{\Gamma(x+S)}{\Gamma(x)} \;=\; \frac{1}{\sqrt{\pi}}\Gamma(S+\frac{1}{2})\,{}_2F_1(2x-1,\frac{S}{2};x+S;\frac{1}{2}). \tag{61}$$

These allow anyone seeking to evaluate $\Gamma(z)$, to reduce z into an arbitrarily small region if we have a sufficiently large table of pretabulated Γ values. For example (if z is real), we can shift z into $-1/6 \le \mathrm{Re}(z) \le 0$, then reflect it about $-1/12$ if necessary to be in $-1/12 \le \mathrm{Re}(z) \le 0$, then reflect it about $-1/24$ if necessary to be in $-1/24 \le \mathrm{Re}(z) \le 0$, then reflect it about $-1/48$ if necessary to be in $-1/48 \le \mathrm{Re}(z) \le 0$, etc (halving the interval each time), provided we have the values of $\Gamma(-2/3)$, $\Gamma(-7/12)$, $\Gamma(-13/24)$, etc, available.

Another idea (which would require a much larger grid-style table of precomputed Γ values – a linear rather than logarithmic number now are required, and indeed a quadratic number if z is complex rather than real) would be to reflect or shift z into $0 \le \mathrm{Re}(z) \le 1/N$, $0 \le \mathrm{Im}(z) \le 1/N$ in just *one* step for some large fixed value of N.

All values $\Gamma(n/24)$ for integer n are known in closed form in terms of algebraic numbers, π, and complete elliptic integral $K(k)$ values computable fast using Gauss's AGM iteration [7][8][21][36], allowing instant range reduction to $0 \le \mathrm{Re}(z) \le 1/48$.

[12] Incidentally, Ramanujan [5] found many spectacular continued fractions for Gamma function ratios, but all his CFs I analysed exhibit very slow *sub*geometric convergence – and in some cases are only claimed to be valid when they terminate (presumably because otherwise they might not converge at all).

Although EQ 60 has problems when when attempting to reflect about 1/4 shifted by half integers (since the Gamma function on its right hand side is infinite) that problem can be sidestepped. Take HOMF 15.1.26 (which is yet another quadratic transformation – HOMF 15.3.31 this time – of a special case of EQ 49) and apply the Γ reflection and cosine angle-doubling formulas to get this additional reflection formula:

$$\Gamma(\frac{1-2M}{4}+x)\Gamma(\frac{1-2M}{4}-x)\cos(2\pi x) \;=\; \frac{2^{1+M}\pi^{3/2}}{M!}\,{}_2F_1(\frac{1}{2}-2x,\frac{1}{2}+2x;1+M;\frac{1}{2}) \tag{62}$$

although[13] admittedly this formula still fails (in the sense that it merely yields $0=0$ or $\infty 0$) if $2x-1/2$ is a nonnegative integer ($x=1/4,3/4,5/4,7/4,\ldots$).

By combining EQs 60 and 51 we find the following representation of the Beta function as a ratio of two exponentially convergent series:

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \;=\; \frac{2}{2a+2b-1}\,\frac{{}_2F_1(1,2a+2b-1;a+b+1/2;1/2)}{{}_2F_1(2a-1,2b-1;a+b-1/2;1/2)}. \tag{63}$$

(Different representations of $B(a,b)$ can be got by combining other of our results.) If we apply the linear transformation HOMF 15.3.3 this may be rewritten in the equivalent form

$$B(a,b) \;=\; \frac{2}{2a+2b-1}\,\frac{{}_2F_1(a+b-1/2,1/2-a-b;a+b+1/2;1/2)}{{}_2F_1(b-a+1/2,a-b+1/2;a+b-1/2;1/2)}. \tag{64}$$

8 New integral representation

Here is a new integral representation of $R(z)$:

Theorem 4.

$$R(z)\stackrel{\text{def}}{=}\frac{\Gamma(z)}{\Gamma(-z)}=-2\int_{-\infty}^{+\infty}\exp([2z+1]u)J_1(2e^u)\mathrm{d}u, \qquad -1<\mathrm{Re(z)}<\frac{1}{4} \tag{65}$$

where J_1 denotes the first Bessel function of the first kind (HOMF ch.9).[14]

Proof of convergence: We first will confirm directly that this integral converges if $-1<\mathrm{Re}(z)<1/4$. The same reasoning also makes it clear it diverges if $\mathrm{Re}(z)\le -1$ or $1/4\le \mathrm{Re}(z)$, and divergence when $z=-1$ is *desirable* since $R(z)$'s first pole occurs at $z=-1$. Observe that when $u\to-\infty$, $J_1(2e^u)\sim e^u$ which is enough to make the integrand decline exponentially there if $-1<\mathrm{Re}(z)$. But by the same argument if $\mathrm{Re}(z)\le -1$ we get *non*convergence due to exponential increase, non-decrease, and/or oscillation. On the other hand when $u\to+\infty$

$$J_1(2e^u)\approx \pi^{-1/2}e^{-u/2}\cos(e^u-3\pi/4), \tag{66}$$

and the ultrafast oscillation is enough to cause the integral – re-expressed as an alternating series "signed bump sum" – to converge ultimately geometrically if $\mathrm{Re}(z)<1/4$. The bumps increase exponentially in height but decline exponentially in width (which is more than enough to compensate, i.e. the bump areas decrease geometrically) if $-1/4<\mathrm{Re}(z)<1/4$; if $\mathrm{Re}(z)=1/4$ the bump areas are asymptotically constant and hence if $\mathrm{Re}(z)\ge 1/4$ the integral does not converge due to oscillation; and if $\mathrm{Re}(z)<-1/4$ then *both* the bump widths and heights decline exponentially, which is a superior kind of convergence. ❄

Remark: Bi-infinite integrals such as this one which decline exponentially at both ends (as this does when $-1<\mathrm{Re}(z)<-1/4$) are numerically desirable because the N-point trapezoid rule converges on them with error declining exponentially with N; it is also enjoyable that the functions J_1 and exp in the integrand are entire and hence have rapidly globally convergent power series.

Proof sketch for validity: Consider the Fourier transform of $R(x+iy)$ with respect to y. This transform may be expressed as a sum instead of an integral by using the Cauchy residue theorem, and then one observes that the sum over residues exactly corresponds to a sum of the same form as the standard globally-convergent power series expansion of

$$J_1(q)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{k!^2}(\frac{q}{2})^{2k-1}, \tag{67}$$

[13] The special case $M=0$ of EQ 62 was exercise 22 p.191 of [7] and they write $K_{2x}(\frac{1}{\sqrt{2}})$ in their notation for the ${}_2F_1$ on our right hand side when $M=0$, where $K_s(k)$ is Ramanujan's generalization of the complete elliptic integral $K(k)$, defined in EQ 5.5.3 page 178 of [7].

[14] `MAPLE`'s symbolic integrator successfully finds $R(0)=-1$. Also `MAPLE`'s *inexact* (numerical) integrator confirms the validity of EQ 65 to ≥ 8 decimal places when $z=-0.25$ and $z=-0.75$, but fails enormously when $z=-0.375$ and $z=-0.625$ where it claims the integral has value 9.6×10^{69} and 2.0×10^{44} respectively! `Mathematica` handles it much better and confirms it both numerically to 5 decimals *and* symbolically in both the latter two cases, as well as correctly finding $R(-1/2)=-2$ (which causes `MAPLE` to core dump) and correctly finding $R(0.2)$ and $R(-0.95)$ to 6 significant figures.

explaining where the J_1 comes from. Finally we take the inverse Fourier transform to get the result. ❄

Remark: The same kind of derivation, but applied to the Gamma function itself (note $\Gamma(z)$ has residues $(-1)^k/k!$ at non-positive integers $z=-k$) instead of $R(z)$, merely leads to an integral equivalent under a change of variables $u=e^w$ to the usual Euler integral (HOMF 6.1.1)

$$\Gamma(z)=\int_0^\infty u^{z-1}e^{-u}\mathrm{d}u=\int_{-\infty}^{+\infty}\exp(wz-e^w)\mathrm{d}w, \qquad \mathrm{Re}(z)>0. \tag{68}$$

The second integral here, although equivalent to the first, is superior in the sense that it is bi-infinite and declining exponentially (or faster) at both ends and hence is better suited to numerical integration via the trapezoidal rule. (The integrand's peak occurs at $w=\ln z$ where it has value $(z/e)^z$; the peak has width of order $z^{-1/2}$.) According to Nielsen ([24] EQ 14 p.145) this alternate form of Euler's integral was recommended by Gauss.

9 More new integral representations

Let $\Gamma(z)=\sqrt{2\pi}z^{z-1/2}\exp(\mu(z)-z)$ where $\mu(z)$ is *Binet's function*, i.e. $\mu(z)=\ln\Gamma(z)-(z-1/2)\ln z+z-\ln\sqrt{2\pi}$. It is well known that $0<\mu(z)<1/(12z)$ if $0<z$. It is known ([2] p24 and [24] EQ 6 p.87; this series is originally due to C.Gudermann in 1845) that

$$\mu(z) \;=\; \sum_{n=0}^{\infty}\left[(z+n+\frac{1}{2})\ln(1+\frac{1}{z+n})-1\right]. \tag{69}$$

This[15] series converges for all complex z with $|\arg z|<\pi$ (which may be seen, as Artin [2] points out on his page 21, because the summand is asymptotic to $(2z+2n+1)^{-2}/3$ when $n\to\infty$).

For each integer $k\ge 0$ we announce the new integral representation

$$\mu(z)=\sum_{n=0}^{k-1}\left[(z+n+\frac{1}{2})\ln(1+\frac{1}{z+n})-1\right]- \tag{70}$$
$$\frac{\pi}{2}\int_{-\infty}^{+\infty}\mathrm{sech}(\pi x)^2\left[\frac{(z-1/2+k+ix)(z+1/2+k+ix)}{2}\ln(1+\frac{1}{z-1/2+k+ix})-\frac{z+k+ix}{2}\right]\mathrm{d}x, \qquad \mathrm{Re}(z)>\frac{1}{2}.$$

The proof is first to apply integration by parts and then Cauchy's residue theorem; the residues then exactly correspond to the summands in Gudermann's series.

We can similarly use

$$\mathrm{HSN}(z)=z\ln(1+\frac{1}{2z})-\frac{1}{2}+\sum_{n=1}^{\infty}\left[(z+n)\ln(1+\frac{1}{z+n-1/2})-1\right], \qquad |\arg(z+1/2)|<\pi, \tag{71}$$

(arising from EQ 10) to get for each integer $k\ge 0$ the new integral representation

$$\mathrm{HSN}(z)=z\ln(1+\frac{1}{2z})-\frac{1}{2}+\sum_{n=1}^{k}\left[(z+n)\ln(1+\frac{1}{z+n-1/2})-1\right]- \tag{72}$$
$$\frac{\pi}{2}\int_{-\infty}^{+\infty}\mathrm{sech}(\pi x)^2\left[\frac{(z+k+ix)(z+1+k+ix)}{2}\ln(1+\frac{1}{z+k+ix})-\frac{z+1/2+k+ix}{2}\right]\mathrm{d}x, \qquad \mathrm{Re}(z)>0.$$

For each integer $k\ge 0$ we also announce the new integral representation

$$\ln\frac{1}{\Gamma(s)} \;=\; (1+\gamma-H_k)s+\ln\left[s\prod_{n=1}^{k}(1+\frac{s}{n})\right]-\frac{\pi}{2}\int_{-\infty}^{+\infty}\mathrm{sech}(\pi x)^2(\frac{1}{2}+k+ix+s)\ln(1+\frac{s}{1/2+k+ix})\mathrm{d}x, \qquad \mathrm{Re}(s)>\frac{-1}{2}. \tag{73}$$

Here $H_k=\sum_{j=1}^{k}1/j$ are the "harmonic numbers." Throughout this paper empty sums are zero so $H_0=0$ (and empty products are 1), so the above formulas both are simplest when $k=0$. The proof is first to apply integration by parts and

[15] Gudermann's series was rediscovered in the exponentiated form

$$n!=\sqrt{2\pi n}(\frac{n}{e})^n\prod_{k=0}^{\infty}\frac{1}{e}(1+\frac{1}{k+n})^{k+n+1/2}$$

140 years later by N.D.Mermin [23]. Because Mermin did not realize this was just Gudermann's series, he failed to realize that his identity is also valid for *non*integer n.

then Cauchy's residue theorem; the residues then exactly correspond to the summands in the series representation of $\ln\Gamma(z)$ that arises from taking the natural log of the Euler-Weierstrass product HOMF 6.1.3.

A few experiments suggest that numerical integration of EQ 73 is a good method of evaluating the log Gamma function, especially if you use k of the same order as both the number of desired decimal places and the number of integration points. These integral representations may be thought of as summing "the rest of the Gudermann series" or "the rest of the Euler product" beyond any point.

One can similarly also turn EQ 47 into an integral valid when $\mathrm{Re}(s), \mathrm{Re}(t) > -1/2$, for representing the log Beta function:

$$\ln\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = \ln\left[\frac{s+t}{st}\prod_{n=1}^{k}\frac{(s+t+n)n}{(s+n)(t+n)}\right] + \frac{\pi}{2}\int_{-\infty}^{+\infty}\mathrm{sech}(\pi x)^2[F_k(s)+F_k(t)-F_k(s+t)]\mathrm{d}x, \tag{74}$$

$$F_k(s) \stackrel{\text{def}}{=} (\frac{1}{2}+k+ix+s)\ln(1+\frac{s}{1/2+k+ix}). \tag{75}$$

Although many other integral representations of $\ln\Gamma(z)$, $\mu(x)$, and the beta function are known (see [24]) none share with these the desirable property of being bi-infinite and exponentially declining at both ends.

10 The failure of Sinc interpolation

Because equality holds in the following when z is an integer, one might imagine that it holds more generally, but it does *not*. The behavior of the right hand side at $z = iy$ far up the imaginary axis is roughly like $c\sinh(\pi y)/y$ for some constant c, whereas by HOMF 6.1.29 the *squared* magnitude of the left hand side is $y\sinh(\pi y)/\pi$. (Also one may confirm inequality numerically at, e.g. $z = 0.3$.)

$$\frac{1}{\Gamma(z)} \neq \frac{\sin(\pi z)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(z-n-1)} \tag{76}$$

In view of that one might then hope for equality in

$$\frac{1}{\Gamma(z)^2} \neq \frac{\sin(\pi z)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!^2\,(z-n-1)} \tag{77}$$

but it still is untrue for the same reason; it is asymptotically off by a factor of order cy^2 far up the imaginary axis, which, while far superior to being off by an *exponentially large* factor, remains imperfect. (And numerical investigation again shows inequality when $z = 0.3$.)

Attempts to correct this by introducing factors like a^2+cz^2 unfortunately also introduce spurious zeros. A different correction attempt is to *add* $z\sin(\pi z)/\pi$ to the right hand side. This causes the *magnitude* of EQ 77 to become asymptotically correct when $z = iy$ with $y \to \pm\infty$, but not its *phase angle* (i.e. arg) because the right hand side is asymptotically pure real but the left hand side is not. Also, again numerically the two sides differ at $z = 0.3$, and this correction term would cause massive disagreement for large real z.

We have mentioned these failures of sinc interpolation as an interesting cautionary note to, e.g, physicists. The failure of the second attempt is actually precisely a borderline case for Carlson's theorem (which we shall discuss next section).

A final (also failed) attempt at sinc interpolation is

$$\frac{1}{\Gamma(z)} \neq \frac{\sin(2\pi z)}{2\pi}\left[\sum_{n=0}^{\infty}\frac{1}{n!\,(z-n-1)} - \sum_{n=-\infty}^{\infty}\frac{1}{(n-1/2)!\,(z-n-1/2)}\right] \tag{78}$$

This one actually might have been justifiable by Carlson's theorem – except for the fact that the second sum (massively) diverges at large negative integers n.

11 Hankel's integral – the original one and a new one

Hankel's contour integral is

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C u^{-z}e^u\mathrm{d}u \tag{79}$$

where C is an anticlockwise contour encircling the negative real axis $(-\infty, 0]$. To prove this, note that $e^x = \sum_{n\geq 0} x^n/n!$ is a generating function for the reciprocated factorial function. Furthermore, the sum may be thought of as extending from $n = -\infty$ to $+\infty$ since $1/n! = 0$ if n is a negative integer. Hence Hankel's integral is immediately seen to be valid if $z = n$ is an *integer*, since it then is simply an instance of Cauchy's residue theorem. However, it is not obvious that it is valid for non-integer z. Fortunately that is justified by F.D.Carlson's 1914 theorem[16] that the only analytic function *periodic* on the real axis that stays *small* as you go up the imaginary axis, is a constant, and considering the fact (a consequence of the reflection formula) that $1/|\Gamma(z)|$ stays exponentially smaller than the Carlson threshold as we travel up lines parallel to the imaginary z-axis (e.g. see HOMF 6.1.29). This 1-paragraph proof is far shorter than previous ones I have heard of. Hankel's original 21-page effort is [14].

Hankel's integral declines exponentially at both ends and hence is numerically friendly to trapezoidal integration. Indeed, it is more numerically desirable than Euler's integral EQ 68 in the sense that it converges for all complex z whereas Euler's integral requires $\mathrm{Re}(z) > 0$.

One equivalent form of Hankel's integral, got by the change of variables $u = (c + ix)^2$, $\mathrm{d}u = 2i(c + ix)\mathrm{d}x$, where c is any positive real constant, is

$$\frac{1}{\Gamma(z)} = \frac{1}{\pi}\int_{-\infty}^{+\infty}(c + ix)^{1-2z}\exp(-[x - ic]^2)\mathrm{d}x. \tag{80}$$

Numerical integration of EQ 79 along the steepest descent contour through the saddlepoint is a possible numerical method for computing $1/\Gamma(z)$. The line of integration in EQ 80 is a fairly good approximation to its own steepest descent contour and an even better approximation would be got by distorting the path of integration in so that for large $|x|$ it gets shifted upward off the real line (x-axis), instead becoming asymptotic to the parallel line $\mathrm{Im}(x) = c$.

There are four special features that made our proof of Hankel's integral work and which made his result desirable:

1. e^x is a generating function for $1/n!$ for *all* integers n, not just the nonnegative ones.
2. e^z has a direction (namely -1) in which it shrinks exponentially.
3. e^z is a function that is simply and rapidly evaluated.
4. The Gamma function $\Gamma(z)$ stays bounded (indeed shrinks exponentially) as one travels up lines parallel to the imaginary axis, and does so sufficiently slowly that Carlson's theorem can be used to justify Hankel's integral for non-integer n.

If one tries to derive a Hankel-formula analogue for $R(z)$, then one can get some of these 4 features, but not all of them at the same time.

In view of $I_0(x) = \sum_{k\ge0}(z^2/4)^k/k!^2$ (HOMF 9.6.10; I_0 is a modified Bessel function) it is simplest to consider $\Gamma(z)^{-2}$ and we would get

$$\frac{1}{\Gamma(z)^2} = \frac{1}{2\pi i}\int_C u^{-z}I_0(2\sqrt{u})\mathrm{d}u \tag{81}$$

where and C is the same sort of contour as before. This integral is absolutely convergent if $\mathrm{Re}(z) > -3/4$ and converges as an alternating "signed bump sum" (but not absolutely) if $\mathrm{Re}(z) > 1/4$. Unfortunately this is a considerably less desirable state of affairs than Hankel's original integral, both because the convergence is only in a halfplane instead of global, and more importantly because the integrand declines only in power-law fashion at both ends, instead of exponentially. The latter criticism can be eliminated by changing variables to rewrite the integral as, e.g.

$$\frac{1}{\Gamma(z)^2} = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}x^{1-2z}I_0(2x)\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{+\infty}(iy + c)^{1-2z}I_0(2iy + 2c)\mathrm{d}y = \frac{1}{\pi}\int_{-\infty}^{+\infty}(c + i\sinh t)^{1-2z}J_0(2\sinh t - 2ic)\mathrm{d}t \tag{82}$$

where the successive changes of variables are $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$, then $x = iy + c$, $\mathrm{d}x = i\mathrm{d}y$, then $y = \sinh t$, $\mathrm{d}y = i\cosh t\mathrm{d}t$ and c is an arbitrary positive real constant. The lattermost integral is bi-infinite and exponentially declining at both ends.

Also, the original integral can be evaluated indefinitely in closed form as a ${}_1F_2$ hypergeometric series by integrating I_0's Maclaurin series term by term:

$$\frac{1}{\Gamma(z)^2} = \frac{1}{(1-z)2\pi i}\lim_{\substack{X\to+\infty\\ \epsilon\to0+}}\left[\frac{X + i\epsilon}{(-X - i\epsilon)^z}\,{}_1F_2(1 - z; 1, 2 - z; -X - i\epsilon) - \frac{X - i\epsilon}{(-X + i\epsilon)^z}\,{}_1F_2(1 - z; 1, 2 - z; -X + i\epsilon)\right] \tag{83}$$

This result can be simplified further by manipulations like the ones we shall employ next section on Hankel's original integral, but seems of comparatively little utility because of the power-law, rather than exponential, decline in the integral's tails which would seem to force calculators to employ very large X.

EQ 81, while plainly valid for *integer* z when convergent (since it was "designed" to be valid because of Cauchy's residue theorem in those cases), cannot be justified using the same argument as before for non-integer z because we now are precisely on the borderline case for Carlson's theorem (i.e. it now is not valid to use that theorem)!

But EQ 81 nevertheless seems valid for non-integer z, because `Mathematica`'s numerical integrator with the 3-line-segment contour $(-1000 - i, 1 - i, 1 + i, -1000 + i)$ and $z = 3.4$ returned the value $0.1125164192197 - 5\times10^{-17}i$, which agrees with $1/\Gamma(3.4)^2 = 0.1125164191929\ldots$ with error 3×10^{-11}.

[16] One precise statement is that if $|f(z)|$ is $O(e^{|kz|})$ in the $\mathrm{Re}(z) \ge 0$ halfplane for some $|k| < \pi$, and if $f(n) = 0$ when $n = 0, 1, 2, 3, \ldots$, then $f(z) \equiv 0$. A proof of this theorem is in [3] pages 36-40.

In view of this, the question of how to prove EQ 81 puzzled me for some time – I was wondering if there were more-precise undiscovered forms of Carlson's theorem. But it eventually dawned that its equivalent form – the first integral stated in EQ 82 – can easily be seen to be valid for *half*-integer $z \ge 1$ by using Cauchy's residue theorem (closing the contour on the parallel line with real part $-c$). That allows us to use the plain old Carlson's theorem but with twice-as-fine periodicity as input (and for *this* purpose, the growth of $1/\Gamma(z)^2$ along the imaginary axis *is* slow enough for Carlson to be applicable) to justify it for non-integer z.

12 Partial Fraction expansions

Let X be real. Truncate Hankel's integral (EQ 79) by integrating it from $-X - i0$ to $-X + i0$ where the truncation becomes exact in the limit $X \to +\infty$. Now employ $e^u = \sum_{n \ge 0} u^n/n!$ and integrate term by term. Also, employ these identities

$$(-X + i0)^p - (-X - i0)^p = 2iX^p \sin(\pi p) \text{ for } X > 0 \quad \text{and} \quad \sin(k\pi + q) = (-1)^k \sin q \text{ for integer } k \tag{84}$$

to simplify the result. We get:

$$\frac{1}{\Gamma(z)} = \frac{-1}{\pi}\Gamma(-z) z \sin(\pi z) \approx \frac{1}{\pi}|X|^{1-z} \sum_{k \ge 0} \frac{(-X)^k \sin([1-z]\pi)}{k!\,(k+1-z)}, \quad X > 0 \tag{85}$$

where the approximation becomes exact (with additive error that declines ultimately exponentially) in the limit $X \to +\infty$. Considering that $\sin([1-z]\pi) = \sin(z\pi)$ and $-\Gamma(-z)z = \Gamma(1-z)$ we may rewrite this by using $s = 1 - z$ as

$$\Gamma(s) = \frac{\pi \csc(s\pi)}{\Gamma(1-s)} = |X|^s \sum_{k \ge 0} \frac{(-X)^k}{k!\,(s+k)} + \frac{E(X)}{\sin(s\pi)}, \quad X > 0 \tag{86}$$

where the error $E(X)$ obeys $|E(X)| \to 0$ exponentially as $X \to +\infty$. It is also possible[17] to derive the same formula more directly just by truncating *Euler's* integral, expanding e^{-u} as a series, and integrating:

$$X^{-z}\Gamma(z) \approx X^{-z} \int_0^X u^{z-1} e^{-u} du = \sum_{k=0}^{\infty} \frac{(-X)^k}{k!\,(z+k)}, \quad \mathrm{Re}(X) > 0, \quad \mathrm{Re}(z) > 0 \tag{87}$$

But our Hankel-based derivation has the advantage that it shows that the approximation to $\Gamma(s)$ is exact in the limit $X \to +\infty$ and exponentially accurate for *all* complex s, whereas Euler's integral only allowed $\mathrm{Re}(z) > 0$. The Euler-based derivation has the countervailing advantage that it gets rid of the $\sin(s\pi)$ factor in the error when $\mathrm{Re}(s) > 0$. The combination of these two facts shows

Theorem 5 (Small errors for the first partial fraction approximation). *Our approximation*

$$\Gamma(s) \approx |X|^s \sum_{k \ge 0} \frac{(-X)^k}{k!\,(s+k)}, \qquad X > 0 \tag{88}$$

(or either of its equivalent forms EQs 92 and 93) to $\Gamma(s)$ has exponentially small (when $X \to +\infty$) relative *and* additive *errors (both) for* every *fixed complex s.*
This also provides another proof of the equality of Hankel's and Euler's integrals.

X	z	rel.err.	abs.err.
20	-45.5	10^{-13}	10^{-10}
10	-5.5	10^{-9}	10^{-5}
10	-2.999	10^{-10}	10^{-5}
20	$5.5i$	10^{-6}	10^{-10}
30	$+5.5$	10^{-6}	10^{-12}

Figure 12.1. Upper bounds on the relative and absolute errors for $\Gamma(z)$ got by using EQ 88 (or equivalently EQ 92 or 93). ▲

Now the sum in EQ 88 *also* can be truncated by summing from $k = 0$ to K, and then we *still* get exponentially small error as $X \to +\infty$ if $K = MX$ for any constant M with $M > e$. To cause the *same* rate of ultimate exponential decrease (namely e^{-X} as $X \to +\infty$ with s fixed) in the errors from both the sum-truncation and the integral truncation, choose $M \approx 3.591121476$ (solving $e^{1/M} = Me^{-1}$). To cause the relative error (and also the additive error, if $\mathrm{Re}(z) \ge 1$) in $\Gamma(z)$ to be below e^{-N}, it suffices if $X > N + (2|\mathrm{Re}(z)| + 3)(\ln|z| + \ln N)$.

[17] And this has been done many times by previous authors well before I was born, perhaps the first being Prym [28] in 1877.

The right hand side of EQ 88 can be directly regarded as an attempted partial fraction expansion of $X^{-z}\Gamma(z)$. Why? It has poles at exactly the same locations – the nonpositive integers $z = -k$ – as does $X^{-z}\Gamma(z)$, and the residues at those poles exactly correspond. Also this right hand side has very pleasant ultimate convergence behavior – it converges, and rapidly, for all complex X and z (excluding poles at nonpositive integer z). However, the right hand side is *not* exactly equal to the left hand side, as is obvious from the fact that the integral (EQ 87) was truncated at X.

Nevertheless, as we've seen, when X is made large, $X^{-z}\Gamma(z)$ and its attempted partial fraction expansion nearly agree, with exponentially small relative error. This is the core of what is, in my opinion, one of the best known algorithms to compute $\Gamma(z)$. But it has deficiencies. One of its least appealing features is this: because the series in EQ 88 is *alternating* and when X is large its sum is much smaller than its individual terms due to massive cancellation, it can be necessary to carry order N extra "guard digits" in an N-decimal-place calculation. For those of us using fixed wordlength floating point arithmetic, that can be a considerable handicap, although for high multiprecision calculations the extra work from carrying the extra guard digits can be comparatively negligible, at least in the right z-domains.

Another reason our attempted partial fraction obviously is not an exact representation is that asymptotically for z large the right hand side tends to zero, whereas the left hand side $X^{-z}\Gamma(z)$ tends to infinity for any fixed X. This gives us the right clue about how to improve things – partial fraction expansions of this sort can *only* hope for exactness if the function they aim to represent, remains *bounded* for large positive z. And indeed

Theorem 6 (Partial fraction expansions). *If $A(z)$ and $B(z)$ are two meromorphic functions whose singularities (1) are in the same locations and (2) each have the same Laurent series as far as all singular terms are concerned, and if $A(z)-B(z)$ grows at most subexponentially along almost all rays emanating from the origin when $|z| \to \infty$, then: $A(z) - B(z)$ is a polynomial in z.*

Proof. $A(z) - B(z)$ is necessarily entire analytic. The result then follows [6] since the only entire analytic functions which grow subexponentially on almost all rays are polynomials. ❄

In the next section we shall prove a much more powerful result about partial fractions, which shall indeed suffice to prove all the results in this section. But for now we shall merely state our results with only heuristic nonrigorous reasoning to justify them.

John **Spouge** [40] introduced a very nice partial fraction type approximation

$$\frac{\Gamma(z)e^{z+N}}{(z+N)^{z-1/2}} = E_N(z) + \sqrt{2\pi} + \sum_{k=0}^{N} \frac{(-1)^k (N-k)^{k+1/2} \exp(N-k)}{k!\,(z+k)} \tag{89}$$

Spouge's approximation becomes exact as the integer N increases to ∞; Spouge proved the relative error term obeys

$$|E_N(z)| < \frac{\sqrt{N+1}}{(2\pi)^{N+1}\,\mathrm{Re}(z+N)} \tag{90}$$

for *all* z with $\mathrm{Re}(z) > -N$ if $N \geq 2$. In practice it empirically appears that the error-decrease factor of $2\pi \approx 6.28$ in Spouge's bound is conservative, and the true factor often seems to be larger. And indeed, we shall see next section that Spouge's approximation is better than he realized: it has *super*exponentially small error, and over a region that includes almost all of the complex plane, not merely a halfplane.

N	z	rel.err.	abs.err.
20	-45.5	huge	1
10	-5.5	10^{-14}	10^{-10}
10	-2.999	10^{-18}	10^{-10}
20	$5.5i$	10^{-26}	10^{-20}
30	$+5.5$	10^{-38}	10^{-29}

Figure 12.2. Upper bounds on the relative and absolute errors for $\Gamma(z)$ for (Spouge's) EQ 89. ▲

Note that this can be interpreted as an *attempt at a partial fraction expansion* of $\Gamma(z)(z+N)^{1/2-z}e^{N+z}$: The *poles* of both the right and left sides have the *same locations* (at $z = 0, -1, -2, \ldots, -N$) and the *same residues*. Furthermore, the left and right hand sides are asymptotic when $|z| \to \infty$ along almost all rays. However, the right side has additional poles at $z = -N-1, -N-2, \ldots$ which do not occur on the left side, which instead has a *branch cut* along the negative real axis for $z < -N$. (This branch cut prevents meromorphicity so that theorem 6 cannot be applied.) Spouge's point is that by putting in these first N poles and replacing the rest with the branch cut, we do not hurt the approximation by much in the right halfplane since all of the "bad things" (branch cuts and ignored poles) are far away and small.

Which is better – Spouge's partial fraction expansion (EQ 89) or our EQ 88? Spouge has the advantage that it has exponentially small *relative* error *everywhere* in the right half plane, and indeed we shall see next section that it does so in the entire complex plane minus a parabola enclosing the $(-\infty, -N]$ real halfaxis. Ours, however, can be regarded as providing exponentially small *additive* error in a *left* half plane (see next section) and hence is less impressive. But Spouge is inferior in the sense that each Spouge term is a more complicated expression. So, for fixed z, our method may be superior, but if $\mathrm{Re}(z)$ is allowed to become large, Spouge's method seems superior. But when $\mathrm{Re}(z)$ becomes *very* large, then the fact that the error in our method (i.e. the chopped-off part of the Euler integral) is exactly expressible as the continued fraction ([20] EQ 11.6 p.144; also see HOMF 6.5.31; originally due to E.Laguerre in 1885)

$$X^{-s}\int_X^\infty u^{s-1}e^{-u}\mathrm{d}u \;=\; \frac{e^{-X}}{X+1-s+}\;\frac{1(s-1)}{X+3-s+}\;\frac{2(s-2)}{X+5-s+}\;\frac{3(s-3)}{X+7-s+\cdots} \tag{91}$$

may give it advantages. This CF converges for all complex X not on the slit $(-\infty,-1]$ with error after n decks ultimately asymptotically proportional to $\exp(-4\sqrt{n})$. That is *subexponential* convergence, but it can converge faster initially than ultimately, e.g. if X is large. The convergents are alternately above and below the limit value for as long as both the partial numerators and denominators stay positive (which is a state of affairs that initially happens if $1 < s < X+1$) which allows us to obtain rigorous upper and lower bounds.

Also, note that the right hand side of EQ 88 can be re-expressed via a Kummer ${}_1F_1$ transformation (HOMF 13.1.27) as[18]

$$X^{-s}\int_0^X u^{s-1}e^{-u}\mathrm{d}u \;=\; {}_1F_1(s-1;s;-X) = e^{-X}{}_1F_1(1;s;X) = e^{-X}\sum_{k=0}^{\infty}\frac{X^k}{s(s+1)(s+2)\cdots(s+k)}. \tag{92}$$

The latter series has the advantage over the original that it (if $X>0$ and at least for all sufficiently large k) is *not* alternating. We also can employ Gauss's continued fraction to rewrite this as[19]

$$= \frac{e^{-X}}{s-X+}\;\frac{1X}{s+1-X+}\;\frac{2X}{s+2-X+}\;\frac{3X}{s+3-X+\cdots} \tag{93}$$

both of these expressions also converge (in the Riemann sphere topology, so we can avoid provisos about the poles at $s=0,-1,-2,\dots$) ultimately superexponentially for *all* complex s and X, and fastest if $|X|$ is small.

Both our and Spouge's expansions unfortunately feature numerically undesirable series alternation and massive cancellation, requiring carrying order N extra "guard digits," but the Legendre and Schlömilch re-expressions we just gave seem superior in the sense that they avoid that bad numerical behavior, e.g. EQ 92 does *not* alternate (at least, not eventually) if $X>0$.

We now enquire: is there a $\Gamma(z)^2$ analogue of these partial fraction approximations, and if so, what advantages does it have? Yes: The $\Gamma(z)^2$ analogue of EQ 88 is

$$X^{-2z}\Gamma(z)^2 \;\approx\; \sum_{k=0}^{\infty}\frac{(-X)^{2k}}{k!^2}\left[\frac{1}{(z+k)^2}+\frac{A(k)/k!-2\gamma-2\ln X}{z+k}\right] \tag{94}$$

where

$$A(0)=0,\quad A(1)=2,\quad A(2)=6,\quad A(3)=22,\quad A(n+2)=(2n+3)A(n+1)-(n+1)^2A(n)\quad\text{for}\quad n\ge 0. \tag{95}$$

Some other expressions for the $A(n)$ are

$$\ln(1-u)^2=\sum_{n\ge1}A(n-1)\frac{u^n}{n!}\quad\text{and}\quad A(n-1)=n!\sum_{k=1}^{n-1}\frac{1}{(n-k)k}\quad\text{for}\quad n\ge 0. \tag{96}$$

(Sequence A052517 in Sloane's encyclopedia [37].) As can be proven from any of the expressions we've given for $A(k)$, most simply the last one, $A(k)/k!{\sim}2\ln(k)/k$ asymptotically for large k so that it has comparatively negligible effect.

In what sense are we to interpret "$\approx$" in EQ 94? By design the two sides have the same poles (same z locations, same singular terms in Laurent series). But they are clearly unequal because when $z\to+\infty$ the left hand side goes to infinity but the right hand side goes to 0. In the limit when $X\to+\infty$ with z fixed, it appears empirically that the approximation becomes exact and the relative error in the approximation of $\Gamma(z)^2$ approaches zero exponentially or faster. See table 12.3. Although at present this is only an empirical finding, a proof will be provided next section.

X	z	rel.err.	abs.err.
20	-45.5	10^{-25}	10^{-137}
10	-5.5	10^{-17}	10^{-21}
10	-2.999	10^{-20}	10^{-16}
20	$5.5i$	10^{-11}	10^{-19}
30	$+5.5$	10^{-15}	10^{-11}

[18] Nielsen ([24] EQ 4 p.28) attributes this to A.M.Legendre in 1811.

[19] Nielsen ([24] EQ 7 p.217) attributes this to O.Schlömilch in 1871, but because it is merely a confluent limit special case of Gauss's continued fraction, it arguably could instead be attributed to Gauss in about 1812.

Figure 12.3. Upper bounds on the relative and absolute errors for $\Gamma(z)^2$ got by using EQ 94. ▲

Although EQ 94 is a more complicated formula than EQ 88, it has the advantage that, for any given X it converges more quickly thanks to the *squared* factorial(k) in the denominator. Thanks to the fact that $(-X)^{2k} = X^{2k}$, it is *not* an alternating series and hence might be hoped to deliver accurate numerical results despite carrying few guard digits.

That hope was my motivation for deriving this, but unfortunately it turns out to be illusory for fixed z with $\mathrm{Re}(z) > 0$, because when $X \to +\infty$ the left hand side approaches zero, and the only way for the right hand side to duplicate that (which it does) is for the $(z+k)^{-2}$ terms in the sum, in aggregate, to nearly exactly cancel the $(z+k)^{-1}$ terms. Hence there is still great cancellation and carrying a large number of guard digits *still*, unfortunately, is necessary.

On the other hand, for fixed z with $\mathrm{Re}(z) < 0$ the left hand side does *not* approach 0 as $X \to +\infty$ so this argument does not apply, and indeed both the $(z+k)^{-2}$ and $(z+k)^{-1}$ terms in the sum then have the *same* sign (namely, positive real part) until k gets larger than $\mathrm{Re}(z)$, and[20] these non-canceled terms *dominate* the sum if $\mathrm{Re}(z) < -X$. For example, when $X = 20$ and $z = -45.5$, where the sum delivered relative error$< 10^{-25}$, cancellation within the sum was negligible and hence almost no extra guard digits were needed.

It is not at all clear which approach – Γ^2 or Γ approximation – is better since the question depends on the details of convergence speed and of the degree of numerical cancellation. We shall not try to decide.

Finally, we construct the $\Gamma(z)^2$ analogue of Spouge's $\Gamma(z)$ approximation,

$$\frac{\Gamma(z)^2\, e^{2z+2N}}{(z+N)^{2z-1}} = E_N(z) + 2\pi + \sum_{k=0}^{N-1} \frac{(N-k)^{2k+1} e^{2N-2k}}{k!^2} \left[\frac{1}{(z+k)^2} + 2\frac{H_k + (N+1/2)/(N-k) - \ln(N-k) - \gamma}{z+k}\right] \tag{97}$$

where $H_k = \sum_{j=1}^{k} 1/j$ are the "harmonic numbers" (and $H_0 = 0$). By design the two sides are asymptotic when $|z| \to \infty$ along almost all rays and their singularities at $z = 0, -1, -2, \ldots, 1-N$ match.

N	z	rel.err.	abs.err.
20	-45.5	10^{-16}	10^{-5}
10	-5.5	10^{-16}	10^{-15}
10	-2.999	10^{-16}	10^{-15}
20	$5.5i$	10^{-41}	10^{-28}
30	$+5.5$	10^{-43}	10^{-62}

Figure 12.4. Upper bounds on the relative and absolute errors for $\Gamma(z)^2$ got by using EQ 97. ▲

The error term $E_N(z)$ may be proven to be superexponentially small as $N \to \infty$, see next section.

13 Partial Fraction expansions (the proofs)

In the last section, we presented many partial-fraction-like approximations for $\Gamma(z)$ or related functions. Only one came with a proof; for the rest we gave only a heuristic derivation or cited the work of Spouge [40]. Spouge's work, however, is long and difficult to digest. We shall now see how all of these results, including a significant strengthening of Spouge's result, can be proven via a simple, unified approach.

Theorem 7 (Partial fraction approximation master theorem). *Let Ω be a closed contour. Let $f(z)$ be a function whose only singularities inside Ω are simple poles. (The theorem and proof are easily generalized*[21] *to handle nonsimple poles of bounded order.) Let them be located at $z = a_1, a_2, a_3, \ldots$ and let the residues at the poles be $b_1, b_2, \ldots$. Then for any point z inside Ω,*

$$f(z) = \sum_m \frac{b_m}{z - a_m} + E, \quad |E| < \frac{1}{2\pi}\left|\int_\Omega \frac{f(w)}{w-z}\mathrm{d}w\right| \tag{98}$$

where the sum is over all poles a_m enclosed by Ω.

Proof: Consider the integral

$$I = \frac{1}{2\pi i}\int_\Omega \frac{f(w)}{w-z}\mathrm{d}w \tag{99}$$

where Ω encloses z. The integrand has poles at the points $w = a_m$ with residues $b_m/(a_m - z)$ and at $w = z$ with residue $f(z)$. Hence by Cauchy's residue theorem

$$I = f(z) - \sum_m \frac{b_m}{z - a_m} \tag{100}$$

[20] The peak summand occurs when $k \approx |X|$.

[21] Simply include higher poles inside the definition of $f(z)$ and use integration by parts when handling those components of the integrand.

precisely gives the additive error in the partial fraction approximation of $f(z)$. ❄

The joy of theorem 7 is that it makes finding good error bounds for many partial fraction type approximations so trivial that one can do it in one's head.

Theorem 8 (Gamma function approximants). *The approximations of EQs 88, 94, 89, and 97 become exact in the limits $X \to \infty$ or $N \to \infty$ with additive error bounds (relative bounds are readily deduced from these) of order* $\exp(-X)$, $\exp(-2X)$, $2^N|\Gamma(1/2-N)|$, $4^N|\Gamma(1/2-N)|^2$ *respectively (up to comparatively unimportant factors at most polynomial in N or X which I have not bothered to compute) if* $\mathrm{Re}(z) < X-1$, *in the first two cases, and in the whole complex plane excluding a standard parabola symmetrically enclosing the interval* $(-\infty, 1/2-N]$ *of the negative real axis, in the last two cases.*

Proofs: ① To (re)prove EQ 88 let Ω be a contour consisting of a vertical line with real part X closed by a radius-$(r+1/2)$ circular arc centered at 0 to its left, where $r \to \infty$ through integer values. The additive error bound that results is of order $\exp(-X)$ (except perhaps for some factors at most polynomial in X which are relatively unimportant when $X \to +\infty$) valid for all z in the halfplane $\mathrm{Re}(z) < X-1$.

② The same contour proves our EQ 94, yielding an additive error bound of order $\exp(-2X)$ (except perhaps for some factors at most polynomial in X) in the same halfplane. Other contours can be used to get even better error bounds albeit in smaller domains.

③ To prove Spouge's EQ 89, let Ω be a contour consisting of a a radius-r circle where $r \to \infty$, minus a parabola symmetrically enclosing the $(-\infty, 1/2-N]$ segment of the negative real axis and of unit curvature near the location $z = 1/2-N$ where it crosses the real axis. The additive error bound that results is of order $2^N|\Gamma(1/2-N)|$ (except perhaps for some factors at most polynomial in N) valid for all z at least distance-1 within the contour. This is because the integral around the full infinite circle is 0 due to the asymptotic exactness of EQ 89 for large $|z|$ in almost all directions (see §4), and the integral along the parabola is of the small order stated.

④ The same Ω works to prove our EQ 97 with an additive error bound of order $4^N|\Gamma(1/2-N)|^2$ (except perhaps for some factors at most polynomial in N). ❄

The partial fraction expansions we and Spouge produced are not really "partial fraction expansions" for the Gamma function because they are only valid in a limiting sense as some parameter approaches infinity. A "genuine" partial fraction expansion such as HOMF 4.3.91-93 is simply an infinite series that returns the function value, with no auxiliary limit being required. Does such an expansion exist for the Gamma function (or some related function such as $E(z)\Gamma(z)^P$ where E is entire and P is a fixed positive integer)? The answer is *no*.

There are two ways to see that. First, an impossibility argument was given by Ritt ([31] p.25) which is completely different from everything here. Second, theorem 7 makes this impossibility clear under the plausible assumption that no entire function $E(z)$ *exists* such that $\int_C E(w)\Gamma(w)^P \mathrm{d}w/(w-z)$ stays small regardless of z in the limit of a very large all-enclosing contour C.

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