Graph-Theoretical Conditions for Inscribability and Delaunay Realizability

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Abstract

We present new graph-theoretical conditions for polyhedra of inscribable type and Delaunay triangulations. We establish several sufficient conditions of the following general form: if a polyhedron has a sufficiently rich collection of Hamiltonian subgraphs, then it is of inscribable type. These results have several consequences:

- All 4-connected polyhedra are of inscribable type.
- All simplicial polyhedra in which all vertex degrees are between 4 and 6, inclusive, are of inscribable type.
- All triangulations without chords or nonfacial triangles are realizable as combinatorially equivalent Delaunay triangulations.

We also strengthen some earlier results about matchings in polyhedra of inscribable type. Specifically, we show that any nonbipartite polyhedron of inscribable type has a perfect matching containing any specified edge, and that any bipartite polyhedron of inscribable type has a perfect matching containing any two specified disjoint edges. We give examples showing that these results are best possible.

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1 Introduction

Delaunay triangulations, and the closely related family of polyhedra of inscribable type, are among the fundamental objects of computational and combinatorial geometry.[1] The problem of providing a graph-theoretical characterization of these structures is a long-standing open problem, dating back to René Descartes [15] and formally posed by Jakob Steiner [27]. A history of the problem and some related results can be found in [18].

Recently, there has been considerable progress on the problem. Jucovič and Ševec have established necessary and sufficient conditions for inscribability in the special case of quadrangular polyhedra satisfying certain constraints on their edge types [21]. Dillencourt [9] has shown that all Delaunay triangulations are 1-tough and have perfect matchings. He has also shown [8] that any outerplanar triangulation is realizable as a Delaunay triangulation or, equivalently, that any pyramid with a triangulated base is of inscribable type (the equivalence follows from Lemma 2.2, below). Rivin [24] has provided a *numerical* characterization of polyhedra of inscribable type as those polyhedra that admit a certain type of weighting (Theorem 2.1, below). Dillencourt and Smith have provided a graph-theoretical characterization of *trivalent* polyhedra of inscribable type, and a linear-time algorithm for recognizing them [12]. Nevertheless, a general graph-theoretical characterization has remained elusive. Examples given in [12] illustrate some of the subtleties involved.

In the present paper, we establish graph-theoretical conditions for inscribability and Delaunay realizability that considerably narrow the gap between the most general sufficient conditions and the strongest necessary conditions. In Section 3 of this paper, we establish several sufficient conditions. Our results say, roughly, that if a planar, 3-connected graph has a sufficiently rich collection of Hamiltonian subgraphs, then it is of inscribable type. These results imply, in particular, that any 4-connected planar graph is of inscribable type (Theorem 3.3), and that any triangulation without chords or non-facial triangles has a combinatorially equivalent realization as a Delaunay triangulation (Theorem 3.6). In addition, we show that any simplicial polyhedron in which all vertices have degrees between 4 and 6, inclusive, is of inscribable type (Theorem 3.9).

In Section 4, we present several necessary conditions for inscribability. In particular, we show that a nonbipartite polyhedron of inscribable type has a perfect matching containing any given edge, and a bipartite polyhedron of inscribable type has a perfect matching containing any two given disjoint edges.

2 Preliminaries

Except as noted, we use the graph-theoretical notation and definitions of [2]. $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. If $S \in V(G)$, $I(S)$ denotes the set of edges incident on some vertex in S, and $N(S)$ denotes the set of all vertices adjacent to some vertex in S. If $v \in V(G)$, $I(v)$ and $N(v)$ are shorthand for

[1] A polyhedron is *of inscribable type* if it has a combinatorially equivalent realization as the convex hull of a set of points on a sphere. This and other terms used in the introduction are defined in Section 2.

$I(\{v\})$ and $N(\{v\})$, respectively. $|S|$ denotes the cardinality of a set S, and $\texttt{deg}(v) = |N(v)|$ denotes the degree of a vertex v. A graph G is *1-tough* [5] if for all nonempty $S \subseteq V(G)$, $c(G - S) \leq |S|$. (Here $c(\cdot)$ denotes the number of connected components.) G is *1-supertough* if, for all $S \subseteq V(G)$ with $|S| \geq 2$, $c(G - S) < |S|$.

A *Hamiltonian cycle* in a graph is a spanning cycle. A graph is *Hamiltonian* if it has such a cycle. A graph is said to be *k-Hamiltonian* if removing any k vertices from it yields a Hamiltonian graph. A k-Hamiltonian graph is $(k+2)$-connected. A famous theorem of Tutte [32, 33] asserts that any 4-connected planar graph is Hamiltonian, and that there is a Hamiltonian cycle passing through any two given edges incident on a common face. A refinement due to Nelson (see [30]) says that any 4-connected planar graph is 1-Hamiltonian. Recently, Thomas and Yu have extended this result by showing that any 4-connected planar graph is 2-Hamiltonian [28].

A *triangulation* is a 2-connected plane graph in which all faces except possibly the outer face are bounded by triangles. The *Delaunay tessellation*, $\mathrm{DT}(S)$, of a planar set of points S is the unique graph with $V(G) = S$ such that the outer face is bounded by the convex hull of S, all vertices on the boundary of a common interior face are cocircular, the vertices of an interior face are exactly the points of S lying on the circumcircle of the face, and no points of S lie in the interior of a circumcircle of any interior face. $\mathrm{DT}(S)$ is said to be *nondegenerate* if it is a triangulation and all convex hull vertices of S are extreme points of S, *degenerate* otherwise. If $\mathrm{DT}(S)$ is nondegenerate, it is called the Delaunay triangulation. Elementary properties of the Delaunay tessellation/triangulation, and the more conventional definition as the dual of the Voronoi diagram, are developed in [1, 13, 23]. We call a triangulation *Delaunay realizable* if it is combinatorially equivalent to a Delaunay triangulation.

A graph G is *polyhedral* if it can be realized as the edges and vertices of the convex hull of a noncoplanar set of points in 3-space (a *polyhedron*). A famous theorem of Steinitz (see [16]) asserts that a graph is polyhedral if and only if it is 3-connected and planar. A polyhedron is *trivalent* if all its vertices have degree 3, *simplicial* if all its faces are triangles. A polyhedron is trivalent if and only if its dual is simplicial. A polyhedron is *of inscribable type* if it has a (combinatorially equivalent) realization as the edges and vertices of the convex hull of a noncoplanar set of points on the surface of a sphere in 3-space. Such a realization is called an *inscription*. Similarly, a polyhedron is *of circumscribable type* if it has a (combinatorially equivalent) realization as a polyhedron each of whose faces is tangent to a common sphere, and such a realization is called a *circumscription*. It is shown in [16] that a polyhedron is of circumscribable type if and only if its dual is of inscribable type.

We will say that a graph is *of inscribable* (respectively, *circumscribable*) *type* if it is polyhedral and if any polyhedron realizing the graph is of inscribable (respectively, circumscribable) type. A *cutset* in a graph is a minimal set of edges whose removal increases the number of components. A cutset is *noncoterminous* if its edges do not all have a common endpoint.

Theorem 2.1 (Rivin [24]; also see [20, 25, 26]) *A graph is of inscribable type if and only if it is polyhedral and weights w can be assigned to its edges such that:*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each vertex v, the total weight of all edges incident on v is equal to 1.*

(W3) *For each noncoterminous cutset $C \subseteq E(G)$, the total weight of all edges in C is strictly greater than 1.*

The following lemma describes the connection between Delaunay tessellations and graphs of inscribable type, using a different formulation from that in [3]. The proof is an immediate consequence of basic properties of stereographic projection [6]. The operation of *stellating* a face f in a plane graph G consists of adding a vertex inside the face f and then connecting all vertices incident on f to the new vertex.

Lemma 2.2 *A plane graph G is realizable as $\mathrm{DT}(S)$ for some set S, with f as the unbounded face, if and only if the graph G' obtained from G by stellating f is of inscribable type.*

The following theorem, which is proved in [11], characterizes the circumstances in which adding edges to graphs of inscribable type preserves inscribability. Here and throughout the paper, we assume that all bipartite graphs are 2-colored red and blue.

Theorem 2.3 ([11]) *Let G be a graph of inscribable type. Suppose that H is obtained from G by performing any of the following transformations in such a way that H remains planar.*

(T1) *If G is nonbipartite, adding an edge to G.*

(T2) *If G is bipartite, adding a red-blue edge to G.*

(T3) *If G is bipartite, adding a red-red edge and a blue-blue edge to G.*

Then H is of inscribable type. Moreover, given any inscription of G, there is an inscription of H that can be realized through an arbitrarily small perturbation of the vertices of the inscription G.

3 Sufficient Conditions

In this section, we establish several sufficient conditions for a polyhedral graph to be of inscribable type. Essentially, our results say that if a planar graph has the property that all the subgraphs obtained in a certain way are Hamiltonian, then the graph is of inscribable type.

We first show that any 1-Hamiltonian planar graph is of inscribable type (Theorem 3.1). This implies, among other things, that any 4-connected planar graph is of inscribable type (Theorem 3.3). Next we show that if a 1-Hamiltonian planar graph satisfies an additional technical restriction then it is Delaunay realizable (Theorem 3.4).

This implies that any 4-connected planar graph can be realized as a Delaunay tessellation (Theorem 3.5), and that any triangulation without chords or nonfacial triangles can be realized as a Delaunay triangulation (Theorem 3.6). We then establish a variant of the 1-Hamiltonian sufficiency theorem (Theorem 3.7), which implies an analogous sufficient condition for bipartite graphs (Theorem 3.8). Finally, we show that if a simplicial polyhedron satisfies certain "near regularity" constraints on its vertex degrees, then it is of inscribable type (Theorem 3.9).

Theorem 3.1 *Any 1-Hamiltonian, planar graph is of inscribable type.*

Proof Let G be 1-Hamiltonian and planar. Since G is 3-connected, it is polyhedral. Let $v_1, \ldots, v_n$ be the vertices of G. For $i \in \{1, \ldots, n\}$, let Z_i be a Hamiltonian cycle through $G - \{v_i\}$. For each $e \in E(G)$, let $x_i(e) = 1$ if Z_i passes through e, 0 otherwise, and let

$$w(e) = \frac{\sum_{i=1}^{n} x_i(e)}{2(n-1)}.$$

Let H be the subgraph of G consisting of those edges e for which $w(e) > 0$. By construction, H is 1-Hamiltonian, hence polyhedral. We claim that the function w, when restricted to $E(H)$, satisfies conditions (W1)–(W3) of Theorem 2.1. Indeed, since each edge e is on at least one and at most $n-2$ of the Z_i, $0 < w(e) < (n-2)/(2(n-1))$, so (W1) is satisfied. Since each vertex of H is on exactly $n-1$ cycles, (W2) holds. Finally, every Z_i crosses each noncoterminous cutset at least twice, so the total weight across each cutset is at least $n/(n-1) > 1$. Hence H is of inscribable type, by Theorem 2.1.

Since H is 1-Hamiltonian, it cannot be bipartite, so adding the edges of $G - H$ to H preserves inscribability by Theorem 2.3. Hence G is of inscribable type. ■

Corollary 3.2 *If $k > 0$, any k-Hamiltonian planar graph is of inscribable type.*

Proof For $k > 0$, any $(k+1)$-Hamiltonian planar graph is necessarily k-Hamiltonian. Indeed, if G is $(k+1)$-Hamiltonian and planar, then G is $(k+3)$-connected, so removing $k-1$ vertices from G leaves a 4-connected graph. Since any 4-connected planar graph is 1-Hamiltonian, it follows that G is k-Hamiltonian. By induction, G is 1-Hamiltonian, hence of inscribable type by Theorem 3.1. ■

Notice that the only feasible values of k in Corollary 3.2 are 1, 2 and 3, since no planar graph can be 6-connected. Corollary 3.2 is false for $k = 0$, as there exist Hamiltonian polyhedra that are not of inscribable type, such as the stellated tetrahedron shown in Figure 1(a).[2] Thomassen has shown that there exist 1-Hamiltonian, planar graphs that are not Hamiltonian [29].

Theorem 3.3 *Any 4-connected planar graph, and any 4-connected planar graph with 1 vertex removed, is of inscribable type.*

[2] The fact that the stellated tetrahedron is not of inscribable type follows immediately from Theorem 3, page 286 of [16] or from Theorem 4.1, below.

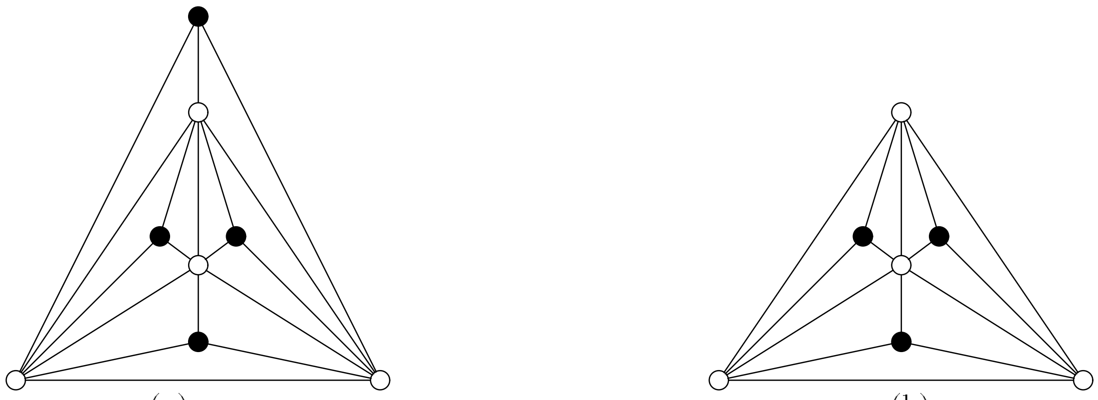

Figure 1: (a) The stellated tetrahedron is Hamiltonian, but not of inscribable type. (b) This graph is 1-Hamiltonian and has a Hamiltonian cycle passing through every edge, but it is not Delaunay realizable.

Proof The first assertion follows immediately from Theorem 3.1 and Nelson's theorem that all 4-connected planar graphs are 1-Hamiltonian. The second assertion follows from Theorem 3.1 and the Thomas-Yu result that all 4-connected planar graphs are 2-Hamiltonian. ■

We note that a 4-connected graph need not be of circumscribable type. Examples are given in [12].

Our next goal is to show that any triangulation without chords or separating triangles is realizable as a Delaunay triangulation (Theorem 3.6). (A *chord* is an edge connecting two nonconsecutive vertices on the outer face, and a *separating triangle* is a nonfacial triangle). We first establish a more general theorem (Theorem 3.4). Before stating this theorem, we remark that it is best possible in the following sense: there exist graphs that are 1-Hamiltonian and have a Hamiltonian cycle passing through every edge but which are not realizable as Delaunay tessellations. One such example is the graph of Figure 1(b), which is not realizable as a Delaunay tessellation because the graph of Figure 1(a) is not of inscribable type.

Theorem 3.4 *If G is planar and 1-Hamiltonian, F is a face of G, and there is a Hamiltonian cycle of G passing through any two consecutive edges on the boundary of F, then G is realizable as a Delaunay tessellation (with outer face F).*

Proof Let G and F be as in the statement of the theorem. Let v_i, $i = 0, \ldots, k-1$, be the vertices of G on the boundary of F, listed consecutively about the boundary of F. Let G' be the graph obtained by stellating face F, with v the stellating vertex. By Lemma 2.2, it suffices to prove that G' is of inscribable type. We construct a weighting of G' satisfying Theorem 2.1 in three steps.

Step 1: Let w be a weighting for G, satisfying conditions (W1)–(W3) of Theorem 2.1. Such a weighting exists by Theorem 3.1.

Step 2: For each $i \in \{0, \ldots, k-1\}$, let Z_i be a Hamiltonian cycle of G using the edges $v_{i-1}v_i$ and $v_i v_{i+1}$, where the subscripts are taken modulo k. For each $i \in \{0, \ldots, k-1\}$ and each $e \in E(G)$, let $y_i(e) = 1/2$ if Z_i passes through e, 0 otherwise. Each function $y_i(\cdot)$ satisfies (W2), and it also satisfies (W1) and (W3) except that the inequalities are not strict. Let

$$y(e) = \frac{w(e) + k\sum_{i=0}^{k-1} y_i(e)}{1+k^2} \tag{3.1}$$

Since y is a convex combination of w and the y_i's, y satisfies conditions (W1)–(W3). Also each edge e incident on F satisfies the inequality

$$y(e) \geq k/(k^2+1) > 1/(2k). \tag{3.2}$$

Step 3: Define a new weighting function x on $E(G')$ by:

$$x(e) = \begin{cases} y(e) & \text{if } e \in E(G) \text{ and } e \text{ is not part of the boundary of } F \\ y(e) - 1/(2k) & \text{if } e \in E(G) \text{ and } e \text{ is part of the boundary of } F \\ 1/k & \text{if } e = vv_i \text{ for some } i \end{cases}$$

It is clear that $x(\cdot)$ satisfies (W1) and (W2). Let C be any cutset in G'. If C contains only edges of G that are not part of the boundary of F, then $\sum_{e\in C} x(e) = \sum_{e \in C} y(e)$. Otherwise, C contains at least one edge incident on v for every pair of edges on the boundary of F, so $\sum_{e \in C} x(e) \geq \sum_{e\in C} y(e)$. Hence $x(\cdot)$ satisfies (W3), and the proof is complete. ■

Theorem 3.5 *Any 4-connected planar graph is realizable as a Delaunay tessellation, with an arbitrary face as its outer face.*

Proof This is immediate from Theorem 3.4, Nelson's Theorem, and Tutte's theorem. ■

Theorem 3.6 *Any triangulation T without chords or nonfacial triangles is realizable as a Delaunay triangulation, with the nontriangular face as the outer face.*

Proof If the outer face has valence 4 or more, then stellating the outer face yields a 4-connected graph, so the result follows from Theorem 3.3 and Lemma 2.2. If the outer face is a triangle, then T is 4-connected, so the result follows from Theorem 3.5. ■

We now turn to bipartite polyhedra. Since no bipartite graph can be 1-Hamiltonian, the preceding theorems do not apply in the bipartite case. Nevertheless, we establish an analog of Theorem 3.1 (Theorem 3.8, below), which is an immediate consequence of the following more general theorem.

Theorem 3.7 *If a planar graph G has the property that removing any pair of adjacent vertices yields a Hamiltonian graph, then G is of inscribable type.*

Proof Let m and n denote the number of edges and vertices of G, respectively. For each edge $e = uv$, let Z_e be a Hamiltonian cycle through $G - \{u, v\}$. Assume, for the moment, that every edge of G lies on at least one of the cycles Z_e. This assumption will be removed at the end of the proof.

For any edges e and e', define $s_e(e') = 1$ if Z_e passes through e', 0 otherwise. We first note that for any vertex $v \in V(G)$,

$$\sum_{e' \in I(v)} \sum_{e \in E(G)} s_e(e') = 2(m - \mathtt{deg}(v)). \tag{3.3}$$

To see that (3.3) holds, reverse the order of summation, and observe that for fixed e, $\sum_{e' \in I(v)} s_e(e') = 0$ if $e \in I(v)$, 2 otherwise.

Next, we note that G is *regularizable* with sum m; that is, there is an assignment $r(\cdot)$ of positive values to edges such that for every $v \in V(G)$,

$$\sum_{e' \in I(v)} r(e') = m. \tag{3.4}$$

Indeed, a regularizing function is given by

$$r(e') = 1 + \frac{1}{2} \sum_{e \in E(G)} s_e(e'). \tag{3.5}$$

This can be seen by summing the right-hand side of (3.5) over all $e' \in I(v)$ and applying (3.3). Observe also that

$$\sum_{e \in E(G)} r(e) = nm/2. \tag{3.6}$$

Now, define a weighting function w on $E(G)$ by

$$w(e') = \sum_{e \in E(G)} \frac{r(e) s_e(e')}{(n-2)m}.$$

We claim that $w(\cdot)$ satisfies conditions (W1)–(W3) of Theorem 2.1. An argument similar to that used to establish (3.3) shows that for any fixed vertex v,

$$\sum_{e' \in I(v)} \sum_{e \in E(G)} r(e) s_e(e') = \sum_{e \notin I(v)} 2r(e) = (n-2)m. \tag{3.7}$$

By (3.7), the total sum of the weights at a vertex v is 1, so (W2) holds.

If $e' = (u, v)$ is an edge, then e' is missed by every cycle Z_e such that either u or v is an endpoint of e. Hence by (3.4) and (3.6),

$$w(e') \leq \frac{nm/2 - 2m + r(e')}{(n-2)m} < \frac{nm/2 - m}{(n-2)m} = 1/2,$$

so (W1) holds.

To show that (W3) holds, let C be a noncoterminous cutset. There are two cases. The first case occurs when one of the components determined by removing C consists of a pair of adjacent vertices. In this case, since (W1) holds, the edge joining them must have weight $< 1/2$. Since (W2) also holds, it follows that C has total weight exceeding 1. In the remaining case, each cycle Z_e must cross C at least twice. Hence, by (3.6) the total weight of the edges in C satisfies

$$\sum_{e' \in C} w(e') = \sum_{e \in E(G)} \frac{r(e)}{(n-2)m} \sum_{e' \in C} s_e(e') \geq \sum_{e \in E(G)} \frac{2r(e)}{(n-2)m} = \frac{n}{n-2} > 1.$$

To complete the proof, we show that the assumption that all edges of G lie on some Z_e is unnecessary. Let H be the subgraph consisting of all edges lying on at least one Z_e. We have just shown that H is of inscribable type. By Theorem 2.3, the only way G could fail to be of inscribable type would be if H were bipartite, G were nonbipartite, and G were obtained from H by adding red-red edges (and possibly red-blue edges) but no blue-blue edges (with respect to an appropriate 2-coloring of H). But in this case, removing two adjacent red vertices from G would create a non-Hamiltonian graph, a contradiction. Hence G is of inscribable type and the proof is complete. ∎

Define a bipartite graph to be *red-blue-Hamiltonian* if whenever a red vertex and a blue vertex are removed, the graph is Hamiltonian.

Theorem 3.8 *If a planar bipartite graph is red-blue Hamiltonian, then it is of inscribable type.*

Proof Immediate from Theorem 3.7. ∎

We conclude this section by showing that if a polyhedron is "almost regular" in a certain sense, then it is of inscribable type.

Theorem 3.9 *Every simplicial polyhedron in which every vertex has degree 4, 5, or 6 is 1-Hamiltonian and hence, by Theorem 3.1, of inscribable type.*

Proof Let G be a simplicial polyhedron in which every vertex has degree 4, 5, or 6. If G is 4-connected, the result follows from Theorem 3.3. So we may assume G is not 4-connected, and hence has a nonfacial triangle, T.

Now consider the possible triplets of numbers of neighbors that the three vertices of T may have inside T, listed in descending order. The possible triplets are: $(1,1,1)$, $(2,1,1)$, $(2,2,2)$, $(3,1,1)$, $(3,2,1)$, $(3,2,2)$, $(3,3,1)$, $(3,3,2)$, and $(3,3,3)$. We may eliminate $(1,1,1)$ as a possibility because it would imply the existence of a vertex inside T of degree 3. It is easy to see that $(2,1,1)$ and $(3,1,1)$ are impossible in a simplicial graph G. It follows (by considering the inside and outside of T simultaneously) that the "complementary" pairs $(3,3,3)$, $(3,3,2)$, and $(3,3,1)$ are also impossible.

Figure 2(a) shows the only way of realizing the triplet $(2,2,1)$; it is not permissible, since there is a vertex of degree 3. This allows us to eliminate the "complementary pair" $(3,2,2)$ as well. Finally, we claim that $(3,2,1)$ cannot be realized. It it were, we would have the configuration shown in Figure 2(b). This configuration has a degree-3 vertex at A, so something must be added inside triangle ABC. Since C already has 5 incident edges, and all triplets containing a 1 except $(3,2,1)$ have already been ruled out, the triangle ABC must have inside it a realization of a $(3,2,1)$ triplet. Repeating the above argument shows that there must be a descending chain of triangles realizing $(3,2,1)$ triplets. Moreover, the chain must continue forever, since if it stops there will be a degree-3 vertex. Since G is a finite graph, this is impossible.

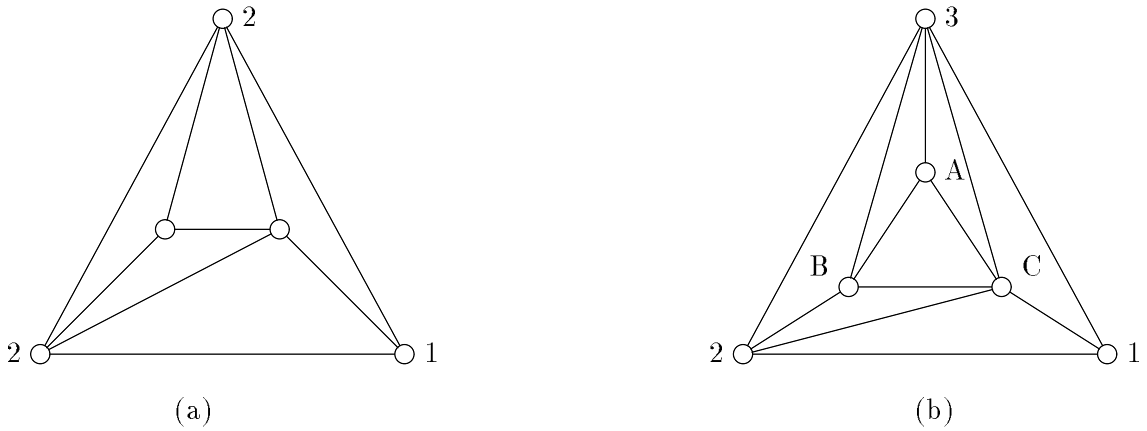

Figure 2: Two nonrealizable degree labelings of triangles.

Thus we have shown that there is only one possibility: each vertex of T must have exactly two neighbors inside T and (by a symmetric argument) two outside. By repeating the argument, it follows that G must be a "string of pearls," a nested sequence of triangles as shown in Figure 3. Such a graph is easily seen to be 1-Hamiltonian. ■

The "string-of-pearls" graphs introduced in the proof of Theorem 3.9 are not 4-connected, but their duals are bipartite and trivalent. It follows from [12, Theorem 3.1] that they are not of circumscribable type. So Theorem 3.9 is false if we replace "inscribable" with "circumscribable". However, we have:

Corollary 3.10 *Every simplicial polyhedron in which every vertex has degree 5 or 6 is of both inscribable and circumscribable type.*

Proof Inscribability is a special case of Theorem 3.9. It follows from the proof of Theorem 3.9 that if G is a simplicial polyhedron in which every vertex has degree 5 or 6, then G is 4-connected. It is observed in [12] that any trivalent polyhedron with a 4-connected dual is of inscribable type (proof: assign each edge a weight of 1/3). This observation implies that G has a dual of inscribable type, so G is of circumscribable type. ■

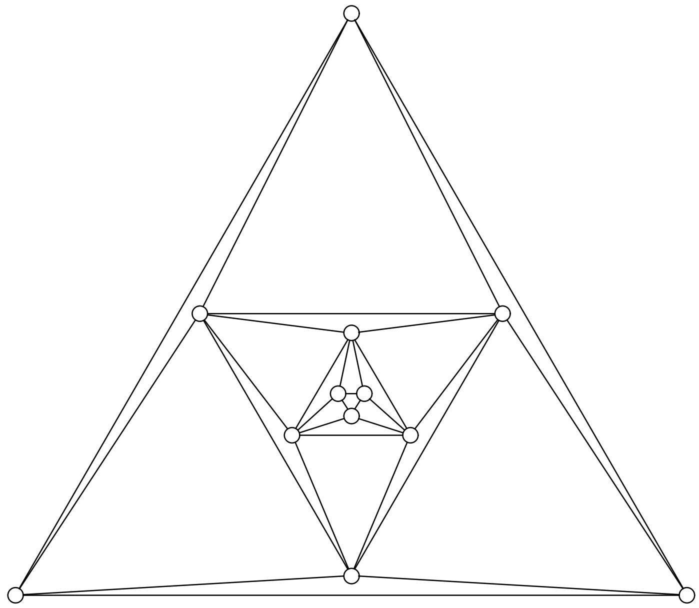

Figure 3: A "string of pearls" graph.

4 Necessary conditions

The following theorem is proved in [12].

Theorem 4.1 *Any nonbipartite graph of inscribable type is 1-supertough.*

The remaining results in this section assert the existence of perfect matchings in polyhedra of inscribable type. A *perfect matching* in an n-vertex graph is a set of $\lfloor n/2 \rfloor$ disjoint edges, where $\lfloor \cdot \rfloor$ denotes the "floor" function. We first state without proof the following lemma, taken from [9], which is an immediate consequence of Tutte's famous characterization of a 1-factor [31]:

Lemma 4.2 *Let G be a graph, and suppose that for each $P \subseteq V(G)$,*

$$c_o(G - P) \leq |P| + 1 \tag{4.1}$$

where $c_o(G - P)$ is the number of components of $G - P$ that have odd cardinality. Then G has a perfect matching.

Theorem 4.3 *Any nonbipartite graph of inscribable type has a perfect matching containing any given edge.*

Proof Remove an edge (and attached vertices) from a nonbipartite graph of inscribable type, and let G be the remaining graph. By Theorem 4.1, (4.1) holds for any subset $P \subseteq V(G)$. Hence G has a perfect matching by Lemma 4.2. ■

Theorem 4.3 is the best possible, in the sense that we cannot always obtain a perfect matching containing two given disjoint edges (or, in the case of a graph with an odd number of vertices, a perfect matching containing a given edge and having a given third vertex as the unmatched vertex). Indeed, consider any graph of inscribable type with an even number of vertices and a nonfacial triangle abc that contains an odd number of vertices in its interior. There is no perfect matching in which a is matched with b and c is matched with a vertex outside triangle abc.

Theorem 4.3 is false if we replace "nonbipartite graph of inscribable type" with "Delaunay triangulation." Indeed, consider any 4 points such that all 4 points are on the convex hull; the (unique) diagonal edge in the Delaunay triangulation cannot participate in a perfect matching.

A stronger version of Theorem 4.3 holds for bipartite graphs of inscribable type:

Theorem 4.4 *Any bipartite graph of inscribable type has a perfect matching containing any two given disjoint edges.*

Proof Suppose that G is bipartite and of inscribable type with $2n$ vertices. Two-color G red and blue. Since all graphs of inscribable type are 1-tough, G has n red vertices and n blue vertices. We claim that any collection of $j \leq n-2$ blue vertices has at least $j+2$ neighbors. This claim implies the theorem. Indeed, let G' be any graph obtained from G by deleting two disjoint edges, the four endpoints of the two edges, and all edges incident on these four endpoints. The claim implies that any collection of j blue vertices in G' has at least j neighbors, so G' has a perfect matching by the Frobenius matching theorem ([22, page 6]).

To prove the claim, let $w(\cdot)$ be a weighting of the edges of G satisfying conditions (W1)–(W3) of Theorem 2.1, and let $j \leq n-2$. Let S be any set of j blue vertices, T its set of neighbors. If $j = 1$, $|T| \geq 3$ since G is 3-connected, so assume $j \geq 2$. The set of edges incident on T but not on S is a cutset. The total weight of this cutset is $|T| - |S|$, an integer. If it is 0, then since G is connected, $|T| = |S| = n$, contradicting the assumption that $|S| \leq n-2$. If $|T| - |S| = 1$, then (W3) implies that the cutset is coterminous, so $|T| = n$, $|S| = n-1$, and the assumption is once again violated. So $|T| - |S| \geq 2$, proving the claim and hence the theorem. ■

Theorem 4.4 is again best possible, as it is not always possible to find a perfect matching containing three given edges. For example, consider the cube: it is easy to select three disjoint edges so that the two unmatched vertices are diametrically opposite vertices. Clearly, these three edges cannot all participate in a perfect matching.

5 Remarks

Theorems 3.1 and 4.1 provide a pair of sufficient and necessary conditions that bracket the class of graphs of inscribable type. Specifically, Theorem 3.1 says that if G is planar and removing any vertex from G yields a Hamiltonian graph, then G is of inscribable type. Theorem 4.1 says that if G is of inscribable type, removing any vertex from G yields a 1-tough graph. It is well known that any Hamiltonian graph is 1-tough [5].

Lemma 2.2 suggests an alternative formulation of these two theorems. Let G be any triangulation with n vertices, and let G' be the simplicial planar graph obtained by stellating the outer face of G. Consider the family $\mathcal{F}$ of $n+1$ triangulations that can be obtained by deleting a vertex of G. Theorem 3.1 says that if *every one* of the triangulations in $\mathcal{F}$ is Hamiltonian, then G' is of inscribable type and hence every triangulation in $\mathcal{F}$ (including G) is Delaunay realizable. Theorem 4.1 says that if G is Delaunay realizable, then every triangulation in $\mathcal{F}$ is 1-tough.

In view of the reformulation in the preceding paragraph, it is tempting to conjecture that there is some graph-theoretical property $\mathcal{P}$, between Hamiltonicity and 1-toughness, such that a nonbipartite polyhedral graph is of inscribable type if and only if removing any vertex produces a graph with property $\mathcal{P}$. A proof of some instantiation of this statement would totally solve Steiner's problem, at least in the nonbipartite case. However, it is not clear what property $\mathcal{P}$ might be.

The existence of a relationship between Hamiltonicity and inscribability has been previously noted. For example, the graphs used to establish an upper bound of $\log_3 2$ on the shortness exponent of simplicial polyhedra in [19] are exactly the same as the graphs used to establish the same upper bound on the noninscribability exponent of simplicial polyhedra in [17]. It was observed in [7] that *any* Hamiltonian polyhedral graph can be inscribed in a certain highly degenerate fashion: the graph can be realized as a polyhedron, "flattened" to a disk, with all the vertices lying on a common circle in an order determined by the Hamiltonian cycle. The results of Section 3 indicate that this relationship is rather strong. Nevertheless, there are limits to the extent of the relationship. In particular, it is an NP-complete problem to determine whether a polyhedron of inscribable type (or a Delaunay triangulation) is Hamiltonian [10].

We close with three open problems:

1. The methods of Section 3 are, in principle, constructive. In particular, a weighting of a 4-connected polyhedron satisfying conditions (W1)–(W3) can be found in quadratic time by repeatedly using algorithms from [4]. Once such a weighting is known, an inscription can be found in polynomial time [24]. Nevertheless, it would be useful to have faster methods for directly constructing inscriptions and Delaunay realizations of these polyhedra.

2. Does removing any pair of adjacent vertices from a bipartite graph of inscribable type leave a 1-tough graph? If so, this would provide a necessary condition for bipartite graphs to be of inscribable type. Such a condition would complement

the sufficient condition of Theorem 3.8, analogous to the complementary relation between Theorems 3.1 and 4.1.

3. Can Theorem 3.9 be extended to include all simplicial polyhedra with 9 or more vertices in which all vertices have degree ≤ 6? The stellated tetrahedron of Figure 1(a), which has 8 vertices, is an example of a simplicial polyhedron with maximum degree 6 that fails to be 1-Hamiltonian. We conjecture that this is the only such example. We have verified this conjecture for all simplicial polyhedra with up to 15 vertices. Ewald has shown that any simplicial polyhedron with maximum degree ≤ 6 is Hamiltonian [14].

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