# Fixed point for negamaxing probability distributions on regular trees

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*Abstract* — **Consider a rooted tree with branching factor $b > 1$ having $b^h$ leafs, each at distance $h$ from the root. Suppose the leafs are assigned real values chosen i.i.d. from some probability density, and the interior nodes of the tree are then also assigned values recursively, according to the *negamax rule*: The value of a node is the maximum of the negated values of its children.**

**The root will then be a random variable, with some probability distribution $\Phi_h(x)$ depending on $b$, $h$, and the distribution $\Phi_0(x)$ of leaf values, and indeed obeying certain recursive relations.**

**We find a closed form for $\Phi_h(x)$. It then turns out that (usually) the behavior of the distribution at the root, when $h$ becomes sufficiently large, is, except for scaling, asymptotically *independent* of the nature of $\Phi_0(x)$ and $h$ and depends only on $b$. The function merely scales self-similarly as $h$ becomes larger. This may be thought of as a new kind of central limit theorem.**

**The limiting form $\Psi_b(x)$, is apparently new to probability theory and is of intrinsic interest as a new special function. Its properties are investigated here, but much remains unknown.**

*Keywords* — Minimax, negamax, Poincare equation, Schröder equation, attractive fixed point, probability distributions, iterative functional equation, special functions, on-line quantile estimates.

## 1 Introduction

### 1.1 Motivation

Trees and the negamax rule arise in the analysis of 2-player, zero sum, alternating move games of perfect information. Namely, each node in the tree represents a game "position," each edge is a "move," and a tree with branching factor $b$ corresponds to a game in which there are $b$ possible moves in each position. The leafs of the tree correspond to terminal positions at which the game is over, and each leaf has associated with it a numerical "payoff" – by convention larger payoffs correspond to game outcomes considered more desirable for the player who did *not* move there. The *negamax rule* associates numerical values to all the interior nodes of the tree (as well as the leafs): The value of a node is the maximum of the negated values of its children. Once the negamaxed node values are known, one may easily find the best move from any given position: namely move to the child with the least value.

Given the fundamental importance of negamaxing in game playing, it is of interest to understand the probabilistic behavior of negamaxed node values. That is the purpose of the present paper.

We will study the simplest case here. Namely we assume that the leaf-payoffs are *independent, identical* random variables with some probability distribution, and that our trees have *constant* branching factor $b > 1$ and *constant* height $h$.

### 1.2 Plan

In section 2, we find a recurrence (EQ 1) allowing us to express the distributions at parent nodes in terms of the distributions of child nodes. We then solve this recurrence in closed form (theorem 1) – but with the aid of a special function we call $\Psi_b(x)$. We then show (theorem 2) that $\Psi_b(x)$ is a *universal attractor*: for a very wide class of leaf-distributions, the distribution at the root will tend to (an appropriately scaled form of) $\Psi_b(x)$.

All the properties of $\Psi_b(x)$ which we could manage to prove, are summarized in section 3. Due to the large number of statements we make about $\Psi_b(x)$, it would have taken too much space to continue beyond section 2 in formal theorem-proof style. Every statement in section 3, however, may be regarded as a theorem.

Rigorous justification of the statements of section 3 will be found in section 4, which comprises the bulk of the paper. There are a few conjectures, also. The most important ones are summarized in section 6 "open problems."

We also include some numerical tables, in section 5, and extensions of our methods to other problems, in section 7.

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### 1.3 Prerequisites and Notation

We assume familiarity with complex analysis [2] [6]. We use the abbreviations "i.i.d." (independently identically distributed) and "CDF" (cumulative probability distribution function). $F^{[n]}(x)$ denotes the result of $n$-time iterative application of function $F$ to argument $x$, while $F^{(n)}(x)$ denotes the $n$th derivative of $F$ at $x$. We use the asymptotic notation $O$, $o$, $\sim$: $f(x) = O(g(x))$ as $x \to \infty$ [or as $x \to 0+$] means "there exists a real constant $c > 0$ such that $f(x) < cg(x)$ for all sufficiently large [or small] positive $x$;" $f(x) = o(g(x))$ as $x \to \infty$ [or as $x \to 0+$] means "for each real constant $c > 0$, $f(x) < cg(x)$ for all sufficiently large [or small] positive $x$;" $f(x) \sim g(x)$ means $\lim f(x)/g(x) = 1$.

## 2 Tree-recurrence, its solution, and a universal attractor

Let $\Phi_h(x)$ be the CDF of the value of a node at height $h$ above the leafs. Thus $\Phi_0(x)$ is the CDF of the leafs themselves. The negamax rule and the assumption of statistical independence implies the recurrence

$$\Phi_h(x) = [1 - \Phi_{h-1}(-x)]^b, \tag{1}$$

valid at all $x$ at which $\Phi_{h-1}(x)$ is continuous. (If $x$ is the abscissa of a jump, (EQ 1) is not valid.)

Upon numerically experimenting with this iteration, one observes that pictures of the graph of $\Phi_h(x)$ tend to settle into a fixed shape as $h \to \infty$, independently of $\Phi_0(x)$, except that the picture of $\Phi_{h+1}(x)$ is *reflected* about the $y$ axis and scaled to be $|s|$ times thinner than the picture of $\Phi_h(x)$, where $|s|$ is a constant depending only on $b$. These numerical experiments motivate us to ask whether there in fact is a CDF $\Psi_b(x)$ which is a *fixed point*, i.e. invariant except for scaling of $x$ by $s$, $s < -1$, of the iteration (EQ 1). Such a $\Psi_b(x)$ would obey

$$\Psi_b(xs) = [1 - \Psi_b(x)]^b. \tag{2}$$

We will later see that solutions of this functional equation do exist, and indeed, if we require that $\Psi_b(x)$ be differentiable and obey $\Psi_b'(0) = 1$, there is a *unique* "principal solution."

From (EQ 2) with $x = 0$, we find (EQ 9) determining $\Psi_b(0)$.

The principal solution obeys

$$\Psi_b'(x) > 0, \quad \lim_{x\to\infty} \Psi_b(-x) = 0, \quad \lim_{x\to\infty} \Psi_b(x) = 1. \tag{3}$$

If we assume, for the moment, that this existence statement and these properties are true (they will be proved later), then we easily obtain the full solution to (EQ 1).

**Theorem 1 [Closed form]** *Let $\Phi_0(y)$ be any continuous CDF and let $\Psi_b(x)$ be the principal solution of (EQ 2), where $s$ is the quantity defined in (EQ 15) and $b > 1$. The general solution of the iterative functional equation (EQ 1) is then*

$$\Phi_h(u) = \Psi_b(s^h \cdot \Psi_b^{-1}[\Phi_0([-1]^h u)]). \tag{4}$$

**Proof.** Existence and uniqueness is obvious. The fact that (EQ 4) is a solution, may be verified directly by induction on $h$. □

We now see why our numerical experiments showed universal asymptotic behavior. The only $x$'s that "matter," as $h$ becomes large, in (EQ 4), are small ones, and the only $y$'s that matter are ones that map to $x$'s with $|x|$ small. More precisely, it is now easy to see that

**Theorem 2 [Universal attractor]** *Let $\Psi_b(x)$ be the principal solution of (EQ 2) where $s$ is the quantity defined in (EQ 15) and $b > 1$. Let $\Phi_0(y)$ be any continuous CDF that is differentiable in a neighborhood of $y = \nu$ such that $\Phi_0(\nu) = \Psi_b(0)$ [given in (EQ 9)], and such that $\Phi_0'(\nu) = c > 0$. Then*

$$\Phi_h([-1]^h[\nu + \frac{x}{s^h c}]) \to \Psi_b(x) \text{ uniformly} \tag{5}$$

*as $h \to \infty$.* □

## 3 A $\Psi_b(x)$ Formulary

In this section we summarize known results about the function $\Psi_b(x)$ which is the "principal solution" of (EQ 2). The proofs of these results will be found in the next section.

All of the results in this section hold if $b > 1$ is any fixed real number (although we have generally thought of it as an integer, previously). Although $\Psi_b(x)$ is an entire analytic function of $x$, we are usually only interested in real $x$.

$\Psi_b^{-1}(q)$ is analytic at $q$ with $q$ real and $0 < q < 1$. $\Psi_b(x)$ is the unique solution of the "functional equation" (EQ 6), which is differentiable at $x = 0$ and satisfies the "standardization condition" (EQ 8), and such a solution exists only if $s$ is given by (EQ 15). $\Psi_b(cx^k)$, $c > 0$, $c$ real, $k \geq 1$, $k$ an odd integer, are the only solutions of (EQ 6) which are differentiable at $x = 0$. Such solutions exist only if $s$ is given by (EQ 62). For further uniqueness results see the last paragraph of section 4.3.

**Functional equation (several equivalent forms):**

$$\Psi_b(xs) = [1 - \Psi_b(x)]^b, \quad \Psi_b(\frac{z}{s}) = 1 - \Psi_b(z)^{1/b}, \tag{6}$$

$$s\Psi_b^{-1}(q) = \Psi_b^{-1}([1-q]^b), \quad \frac{1}{s}\Psi_b^{-1}(y) = \Psi_b^{-1}(1 - y^{1/b}). \tag{7}$$

**Standardization:**

$$\Psi_b'(0) = 1. \tag{8}$$

**$\Psi_b(0)$ and the scaling factor $s$:**

$$\Psi_b(0) = \text{the unique real root of } (1-x)^b = x \text{ in } (0,1). \tag{9}$$

$$\frac{1}{b+1} < \Psi_b(0) < \frac{1}{2}. \tag{10}$$

$$\frac{\partial}{\partial b}\Psi_b(0) = \frac{[1-\Psi_b(0)]\Psi_b(0)\ln[1-\Psi_b(0)]}{(b-1)\Psi_b(0)+1} < 0. \tag{11}$$

$$\Psi_b(0) = \frac{\ln b - \ln\ln b + \ln\ln\ln b - \ldots \pm \ln^{[n]} b + O(\ln^{[n+1]} b)}{b} \tag{12}$$

for $n = 1, 2, 3 \ldots$ as $b \to \infty$. In fact, apparently

$$\Psi_b(0) \in [\frac{1}{1+\epsilon}, 1+\epsilon] \cdot \frac{\ln b - \ln\ln b}{b} \tag{13}$$

for all $b \geq 2.7$, when $\epsilon = 0.124$.

$$\frac{1-2\Psi_b(0)}{L} = a - a^2 - \frac{(-6-3L+2L^2)a^3}{6} + \left(-1 - \frac{3L}{2} + L^2\right)a^4 + \frac{(60+180L-90L^2-35L^3+8L^4)a^5}{60} - \frac{(12+60L-10L^2-35L^3+8L^4)a^6}{12} + \ldots \tag{14}$$

where $L = \ln 2$, $b = 1 + 2a$, $|a| < 1/2$.

$$s = \frac{-b \cdot \Psi_b(0)}{1 - \Psi_b(0)}, \tag{15}$$

$$s \sim -\ln b \text{ as } b \to \infty, \tag{16}$$

$$-b < s < -1, \quad 1 < s^2 < b. \tag{17}$$

$$\frac{s+1}{L} = \left(2 - \frac{2}{L}\right)a + (2-2L)a^2 + \left(-2 + L + \frac{4L^2}{3}\right)a^3 + \frac{6+3L-10L^2-2L^3}{3}a^4 + \left(-2 - 4L + 5L^2 + \frac{19L^3}{6} + \frac{4L^4}{15}\right)a^5 + \frac{90+360L-195L^2-420L^3-81L^4-4L^5}{45}a^6 + \ldots \tag{18}$$

where $L = \ln 2$, $b = 1 + 2a$, $|a| < 1/2$.

**Limiting special values:**
Let $\Psi_1(x) = \lim_{b\to 1^+} \Psi_b(x)$, $\Psi_1'(x) = \lim_{b\to 1^+} \Psi_b'(x)$. Then

$$\Psi_1(z) = z + \frac{1}{2} \text{ if } |z| \leq \frac{1}{2}, \quad \Psi_1(z) = \frac{\operatorname{sign}(z)+1}{2} \text{ otherwise.} \tag{19}$$

$$\Psi_1'(z) = \begin{cases} 1 \text{ if } |z| < 1/2 \\ 0 \text{ otherwise.} \end{cases} \tag{20}$$

**Monotonicity:**

$$\Psi_b'(x) > 0, \quad \lim_{x\to\infty} \Psi_b(-x) = 0, \quad \lim_{x\to\infty} \Psi_b(x) = 1. \tag{21}$$

**Maclaurin series:**

$$\Psi_b(z) = \sum_{n\geq 0} f_n(b) z^n, \quad |z| < \infty, \tag{22}$$

$$f_0 = \Psi_b(0), \quad f_1 = 1, \tag{23}$$

Let $g = 1 - f_0 = (f_0)^{1/b} = -bf_0/s$; then:

$$\frac{f_2}{f_0} = \frac{\binom{b}{2}}{bf_0 g + g^2 s^2} > \frac{b-1}{2(b+1)} > 0, \tag{24}$$

$$\frac{f_3}{f_0} = \frac{-\binom{b}{3} + 2\binom{b}{2} f_2 g}{bf_0 g^2 + g^3 s^3}, \tag{25}$$

$$\frac{f_4}{f_0} = \frac{\binom{b}{4} - 3\binom{b}{3} f_2 g + \binom{b}{2}({f_2}^2 + 2f_3)g^2}{bf_0 g^3 + g^4 s^4}, \tag{26}$$

$$\frac{f_5}{f_0} = \frac{-\binom{b}{5} + 4\binom{b}{4} f_2 g - 3\binom{b}{3}({f_2}^2 + f_3)g^2 + 2\binom{b}{2}(f_2 f_3 + f_4)g^3}{bf_0 g^4 + g^5 s^5}, \tag{27}$$

$$\frac{f_6}{f_0} = \frac{\left[\begin{array}{c} \binom{b}{6} - 5\binom{b}{5} f_2 g + \binom{b}{4}(6{f_2}^2 + 4f_3)g^2 \\ -\binom{b}{3}({f_2}^3 + 6f_2 f_3 + 3f_4)g^3 \\ +\binom{b}{2}({f_3}^2 + 2f_2 f_4 + 2f_5)g^4 \end{array}\right]}{bf_0 g^5 + g^6 s^6}, \tag{28}$$

$$\frac{f_7}{f_0} = \frac{\left[\begin{array}{c} -\binom{b}{7} + 6\binom{b}{6} f_2 g - \binom{b}{5}(10{f_2}^2 + 5f_3)g^2 \\ +\binom{b}{4}(4{f_2}^3 + 12 f_2 f_3 + 4f_4)g^3 \\ -3\binom{b}{3}({f_2}^2 f_3 + {f_3}^2 + 2f_2 f_4 + f_5)g^4 \\ +2\binom{b}{2}(f_3 f_4 + f_2 f_5 + f_6)g^5 \end{array}\right]}{bf_0 g^6 + g^7 s^7}. \tag{29}$$

$$f_n = \frac{\binom{b}{n} f_0}{(-s)^n} \cdot [1 + O(\frac{1}{|s|})], \quad n \text{ fixed}, \quad b \to \infty. \tag{30}$$

$f_n$ is positive for $n$ fixed, for all sufficiently large $b$. But since $\Psi_b(x)$ is bounded, there are an infinite number of negative $f_n$ for any fixed $b$.

$$|f_n| \leq n^{-\frac{\log b}{\log |s|} n + o(n)} \quad b \text{ fixed}, \quad n \to \infty. \tag{31}$$

**Expression of $\Psi_b^{-1}(q)$ as limit of an iterative process:**

$$\text{Let } q_0 = q, \quad q_n = 1 - (q_{n-1})^{1/b}, \quad n \geq 1. \tag{32}$$

$$\text{Then } \Psi_b^{-1}(q) = \lim_{n\to\infty} s^n [q_n - \Psi_b(0)]. \tag{33}$$

$$\text{Let } r_0 = 1, \quad r_n = \frac{-(q_{n-1})^{(1-b)/b} r_{n-1}}{b}, \quad n \geq 1. \tag{34}$$

$$\text{Then } \frac{\partial}{\partial q}\Psi_b^{-1}(q) = \prod_{m\geq 0} \frac{-s \cdot (q_m)^{(1-b)/b}}{b} = \lim_{n\to\infty} s^n r_n > 0. \tag{35}$$

$$\text{Let } t_0 = 0, \quad t_n = \frac{1-b}{b}\frac{r_n r_{n-1}}{q_{n-1}} + \frac{r_n t_{n-1}}{r_{n-1}}, \quad n \geq 1. \tag{36}$$

$$\text{Then } \frac{\partial^2}{\partial q^2}\Psi_b^{-1}(q) = \lim_{n\to\infty} s^n t_n. \tag{37}$$

Let $u_0 = 0$ and

$$u_n = \frac{1-b}{b}\frac{(r_{n-1}t_n + r_n t_{n-1})q_{n-1} - (r_{n-1})^2 r_n}{(q_{n-1})^2} \tag{38}$$

$$+ \frac{(r_n u_{n-1} + t_{n-1} t_n) r_{n-1} - (t_{n-1})^2 r_n}{(r_{n-1})^2}, \quad n \geq 1; \tag{39}$$

$$\text{Then } \frac{\partial^3}{\partial q^3}\Psi_b^{-1}(q) = \lim_{n\to\infty} s^n u_n. \tag{40}$$

Note: The following formulae are useful when using the previous formulae. Let $q(x)$ be a 3-times differentiable function with an inverse. Then

$$\frac{dx}{dq} = \frac{1}{q'}, \tag{41}$$

$$\frac{d^2x}{dq^2} = \frac{-q''}{(q')^3}, \tag{42}$$

$$\frac{d^3x}{dq^3} = \frac{3(q'')^2 - q'q'''}{(q')^5}. \tag{43}$$

**Expression of $\Psi_b(x)$ as limit of an iterative process:**
Define $T_b(x) = (1-x)^b$. Then for any fixed integer $k \geq 1$,

$$\Psi_b(x) = \lim_{n\to\infty} T_b^{[n]}[\sum_{m=0}^{k} f_m(b) \cdot (\frac{x}{s^n})^m]. \quad (44)$$

$$\Psi'_b(x) = \prod_{n\geq 1} \frac{-b}{s}[1 - \Psi_b(\frac{x}{s^n})]^{b-1}, \quad |x| < \infty. \quad (45)$$

**Concavity properties:** The following functions of $x$ are concave-down for all real $x$:

$$\log[1 - \Psi_b(x)], \ \log \Psi_b(x), \ \text{and} \ \log \Psi'_b(x). \quad (46)$$

**Mode, Inflection points:** It is known that $\Psi'_b(x)$ is unimodal and that the mode occurs at a positive $x$-coordinate. It is conjectured that it has exactly 2 inflection points for each $b$.

If $M = x$ is the mode, then

$$L < \Psi_b(M) < U = 1 - (1 - L^b)^b \quad (47)$$

where $L = 1 - \kappa$ and $\kappa$ is a root of $(1 - \kappa^b)\kappa = (s^2/b^2)^{1/(b-1)}$. As $b \to \infty$,

$$L \sim \frac{2\ln b}{b} + \frac{2\ln\ln b}{b} \sim 2\Psi_b(0), \quad 1 - U \sim \frac{(\ln b)^2}{b}. \quad (48)$$

**Moments (which exist for all nonnegative integer $k$):**

$$\mu_k = \int_{-\infty}^{\infty} x^k \Psi'_b(x)dx = \int_0^1 [\Psi_b^{-1}(q)]^k dq \quad (49)$$

$$= k\int_0^{\infty} x^{k-1}[1 - \Psi_b(x) + \Psi_b(-x)]dx \quad (50)$$

$$= k\int_0^{\infty} x^{k-1}[\Psi_b(xs)^{1/b} + \Psi_b(-x)]dx \quad (51)$$

It is known that $\mu_k > 0$ for nonnegative integer $k$. See (EQ 93) for a stronger statement.

$$\mu_k < [1 + \delta_{k \text{ even}} s^2][1 - \Psi_b(0)]^{1+k} k!, \quad k > 0. \quad (52)$$

The behavior as $b \to \infty$ is unknown and I only have crude bounds at present.

**Inequalities:** Let $\xi = \Psi_b(0)$. Then

$$\Psi_b(x) \leq \xi e^{x/\xi}, \quad 1 - \Psi_b(x) \leq (1 - \xi)e^{-x/(1-\xi)}, \quad (53)$$

$$\Psi_b(xy) < \frac{\xi\Psi_b(x)}{(1 - y)\Psi_b(x) + y\xi}, \quad x < 0, \ y > 0, \quad (54)$$

$$\Psi_b(xy) > \frac{[\Psi_b(x) - \xi]y - [\Psi_b(x) - 1]\xi}{[\Psi_b(x) - \xi]y \ - \ [\Psi_b(x) - 1]}, \quad x > 0, \ y > 0. \quad (55)$$

[In the latter two inequalities, equality occurs iff $y = 1$ or $x = 0$.]

$$\Psi_b(x) < \frac{\xi^2}{\xi - x}, \quad x < 0, \quad (56)$$

$$\Psi_b(x) > \frac{x - (\xi - 1)\xi}{x - (\xi - 1)}, \quad x > 0. \quad (57)$$

$$[\Psi'_b(u) - \Psi'_b(su)] \cdot u \geq 0. \quad (58)$$

$$\Psi'_b(-x) < \Psi'_b(x), \ x > 0. \quad (59)$$

**Asymptotics:** See section 4.4.

**Maximum entropy principle:** It is unknown if $\Psi_b(x)$ obeys one. See open problem 4.

**Behavior as a function of $b$:** $\Psi_b(x)$ is an analytic function of $b$ in some complex neighborhood of each fixed real number $b > 0$, where $x$ is fixed. $\Psi_b(0)$ is analytic for all complex $b$ except for $b = 0$.

## 4 PROOFS.

We propose to begin with the fundamental functional equation (EQ 2) and to study properties which any solution must have. We will always assume $b > 1$ and $|s| \neq 0, 1$. We will sometimes depend on smoothness and continuity assumptions. Eventually we will prove *existence* of a solution satisfying all the smoothness assumptions we ever wanted, thus demonstrating that our previous elucidation of properties, was not vacuous.

First, since $\Psi_b(x)$ is a CDF,

$$\begin{gathered}\Psi_b(x) \text{ is monotonic increasing;} \\ \lim_{x\to\infty}\Psi_b(-x) = 0, \ \lim_{x\to\infty}\Psi_b(x) = 1.\end{gathered} \quad (60)$$

Due to the monotonicity, $\Psi_b(x)$ has an inverse, $\Psi_b^{-1}(q)$, for all real $q$, $0 < q < 1$. This justifies the re-expression of the functional equation in any of the alternative forms (EQ 6) (EQ 7).

The expression (EQ 9) for $\Psi_b(0)$ follows from (EQ 2) and the requirement that it lie in $[0, 1]$. (The fact that the polynomial equation has a unique root in $[0, 1]$ is obvious since the left hand side is continuously decreasing from 1 to 0 and the right hand side is continuously increasing from 0 to 1.) The expression (EQ 11) then follows by implicit differentiation of (EQ 9), and (EQ 12) may be verified by direct substitution into (EQ 9). $\frac{\partial}{\partial b}\Psi_b(0) < 0$ is then manifest, which in turn justifies $\Psi_b(0) < 1/2$. From a local analysis near $a = 0$ we find (EQ 14), which may be verified by direct substitution into (EQ 11); (EQ 18) is a direct consequence. [The fact that the radius of convergence of these series is $1/2$ will be shown in section 4.9.]

### *4.1 Arguments depending on various smoothness assumptions*

There is no question that solutions of (EQ 2) exist. In fact, an uncountably infinite number of them exist for each choice of $s$ and $b$. Namely, take any function $f(x)$ you like, and define $\Psi_b(x)$ to be $f(x)$ if $1 \leq x < |s|$, and to be obtained by application of (EQ 2) otherwise. The result obeys (EQ 2) everywhere. If we now restrict our choice of $f$ so that $\Psi_b(0) \leq f(x) < 1$ and so that $f(x)$ is monotonically increasing for $1 \leq x < s^2$, and $(1 - [1 - f(1)]^b)^b \geq f(1)$, then (EQ 60) will also be satisfied.

(We have been assuming $s < -1$, but if $0 < s < 1$ or $-1 < s < 0$, etc, similar constructions also work.) All this is very boring and of no real use.

However, if we ask also that our solutions satisfy certain *smoothness properties*, it will turn out that then, there are much fewer of them. The general solution we just constructed, unless a miracle occurs, will be rough near $x = 0$. (By starting with smooth $f(x)$, though, we can always force it to be smooth at all real *nonzero* $x$.) Indeed we will see that smooth solutions can only exist if $s$ lies in a certain set, and the more smoothness properties we require, the less solutions there are, and indeed, it is far from immediately clear that any *analytic* solution (for example) exists at all. Later, we will answer such worries by producing an existence proof which will tell us exactly when analytic solutions exist, plus a good deal more. For the moment, however, let us argue about what properties suitably smooth solutions must have, assuming that they exist.

Observe that solutions $\Psi_b(x)$ of (EQ 2) have *scaling freedom*, i.e., if $\Psi_b(x)$ is a solution of (EQ 2) (EQ 3), then so is $\Psi_b(cx)$ for any $c > 0$. Also observe that $\Psi_b(x)$ has *power-law freedom*: if $\Psi_b(x)$ is a solution of (EQ 2) (EQ 3), then so is $\Psi_b[|x|^k \operatorname{sign}(x)]$, where $k$ is any positive real, except that the "$s$" in (EQ 2) is different – it has to be replaced by $\operatorname{sign}(s)|s|^{1/k}$.

Let us now assume that $\Psi_b(x)$ is *k-times differentiable* in a neighborhood of $x = 0$, and $\Psi_b^{(k)}(0) > 0$ but $\Psi_b^{(m)}(0) = 0, m = 1, 2, \ldots, k-1$. Of course, necessarily $k$ will be odd, since we only allow monotonic solutions. We will later see that, due to power-law freedom, we may assume $k = 1$ wlog; nevertheless, for now, continue to consider general odd positive $k$.) By differentiating (EQ 2) $k$ times, we obtain

$$s\Psi_b^{(k)}(xs) = -[1 - \Psi_b(x)]^{b-1} \cdot b \cdot \Psi_b^{(k)}(x) \qquad (61)$$

and then, substituting $x = 0$ and using (EQ 2) again gives a formula for $s$:

$$s = -\left[\frac{b \cdot \Psi_b(0)}{1 - \Psi_b(0)}\right]^{\frac{1}{k}} < -1. \qquad (62)$$

If $s$ does not obey this formula, then no solution of (EQ 2) satisfying our assumptions re differentiability at $x = 0$, can exist. The fact that $s < -1$ is equivalent to the fact that $\frac{1}{b+1} < \Psi_b(0)$, which arises from the standard inequality

$$(\frac{b}{b+1})^b > \frac{1}{e} > \frac{1}{b+1}, \quad b > e - 1 \approx 1.71828. \qquad (63)$$

(In fact, $s < -1$ for all real $b > 1$, as may be shown with slightly more work.) The fact that $\Psi_b'(x) > 0$ for all real $x$, assuming that this is true at $x = 0$ and $\Psi_b'(x)$ exists everywhere, follows from (EQ 61).

Let us now suppose that $\Psi_b(x)$ is infinitely differentiable at $x = 0$. In this case, it has a Maclaurin series expansion as in (EQ 22) whose truncation after the $f_k x^k$ term approximates $\Psi_b(x)$ to error $o(x^k)$ as $x \to 0$. We may then solve for the coefficients $f_n(b)$, term by term, by the use of (EQ 2). We already know $f_0(b) = \Psi_b(0)$, see (EQ 9), and we have assumed by fiat that $f_1 = f_2 = \ldots = f_{k-1} = 0$ and $f_k > 0$. In fact $f_k$ is totally free, subject to the restriction $f_k > 0$, due to scaling freedom. For this reason, we may and will demand, without any loss of generality, that $f_k(b) = 1$. The $f_n$ for $n > k$ may be obtained as rational functions of $s$, $b$, and $f_m$ with $k < m \le n$ (where $f_k$ is regarded as a free parameter; $s$ and $f_0$ are of course known functions of $b$) by direct substitution into (EQ 2). The expressions (EQ 24)-(EQ 30) arise in this way with $k = 1$, $f_1 = 1$.

Now, let us further assume that this Maclaurin series actually *converges* to $\Psi_b(x)$ for all $x$ with $|x| < r$, for some $r > 0$. This convergence, plus the fact that $s < -1$, plus the recurrence (EQ 2), shows that $\Psi_b(x)$ is then uniquely determined for *all* real $x$ by its power series coefficients, which in turn are determined uniquely once $k$ and $f_k$ are specified.

We now see that by power-law freedom plus uniqueness, we might as well, without loss of generality, require $k = 1$, and as we've already stated, scaling freedom allows us to choose $f_1 = 1$. This justifies, for example, (EQ 8) and (EQ 15).

Let us now consider the radius of convergence of the Taylor series (EQ 22) in the complex plane. We extend the definition of $\Psi_b(x)$ into the complex $x$-plane by requiring (EQ 2) to hold for complex $x$ where the series converges and also by defining $\Psi_b(x)$, for $|x|$ sufficiently small, to be whatever the Taylor series (EQ 22) converges to. We may do this since it will not affect the behavior of $\Psi_b(x)$ for $x$ real; and since we are only making our definitions within the region of convergence, we have not lost any generality.

Now, if there is any point $z$ of nonanalyticity of $\Psi_b(z)$ (at which $\Psi_b$ or any of its derivatives become infinite) then (EQ 2) tells us that there is another such point at $z/s$, and at $z/s^2$, etc. – contradicting our assumption that the Taylor series converged to $\Psi_b$ in "some neighborhood of $x = 0$." We therefore see that this assumption forces $\Psi_b(z)$ to be an *entire* function – the series (EQ 22) converges everywhere.

Indeed, referring to standard theorems and definitions about entire functions in chapters 1 and 2 of [1], we see that $\Psi_b(z)$, $z$ complex, has "order"

$$\rho = \limsup_{|z|\to\infty} \frac{\log\log|\Psi_b(z)|}{\log|z|} = \frac{\log b}{\log|s|} > 1, \qquad (64)$$

from which it follows that

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/|f_n|)} = \rho, \qquad (65)$$

justifying (EQ 31). If $\rho$ is noninteger, then $\Psi_b(z)$ has an infinite number of complex zeros (or for that matter, an infinitely large number of assumptions of any fixed complex value). Indeed, if $N(r)$ denotes the number of zeros within $|z| < r$, we have (assuming $\rho$ is noninteger)

$$\lim_{r\to\infty} \frac{\log N(r)}{\log r} = \rho. \tag{66}$$

The integrality or non-integrality of $\rho$ seems to be irrelevant in our particular case, so presumably this conclusion still holds when $\rho$ is noninteger.

Incidentally, when working in the complex plane, beware of branch points. The two "backward" forms of the functional equations (EQ 6) (EQ 7) [involving $b$th roots] are no longer unambiguous, as they previously were when their arguments lay in the real interval $(0,1)$. Thus, one might erroneously conclude that there are an infinite number of zeroes of $\Psi_b(z)$ within $|z| < r$ for any $r > 0$, since if $z$ is a zero, "then" so is $z/s^2$. Not so! On the other hand, our previous argument that "if $z$ is a point of nonanalyticity, where some [and hence all succeeding] derivative of $\Psi_b(z)$ becomes infinite, then so is $z/s$," is immediately justifiable, unless $\Psi_b(z) = 0$, in which case it is (slightly less immediately) justifiable when $b > 1$.

So, to summarize what we have discovered so far: We've now shown that either *no* solution of (EQ 2) exists which has Maclaurin series expansion convergent to it in some neighborhood of $x = 0$, or else all such solutions are variants, obtained by scaling and power-law freedom, of exactly one "principal solution." This principal solution, should it exist, is an entire analytic function with all kinds of properties we already know. It is now getting rather embarrassing that we have not yet proved existence.

### *4.2 An iterative process and an existence proof*

To show existence, we analyse the iterative process (EQ 32) (EQ 33). This process may be rephrased as follows. Define

$$U(x) = 1 - x^{1/b}. \tag{67}$$

(We should perhaps have called this $U_b(x)$, but we suppress the dependence on $b$ since we will regard $b$ as a fixed constant throughout the proof.) Then (we claim)

$$\Psi_b^{-1}(q) = \lim_{n\to\infty} s^n [U^{[n]}(q) - \Psi_b(0)]. \tag{68}$$

First, we observe that if the limit in (EQ 68) exists, then it must be a solution of the functional equation (EQ 7): $\frac{1}{s}\Psi_b^{-1}(q) = \Psi_b^{-1}[U(q)]$.

Second, we observe that the iteration-function $U(x)$ maps the interval $I = (0,1)$ to itself and has a unique fixed point $x = F$, at $F = \Phi_b(0)$, in $I$. In fact, this fixed point is a *global attractor* – $\lim_{n\to\infty} U^{[n]}(x) = F$ for all $x \in I$. This fact follows after seeing that $U[U(x)] = x$ has exactly one root, namely $x = F$, for $x \in I$. $U[U(x)] - F$ and $x - F$ always have the same sign, for $x \in I$, but the former quantity has smaller magnitude (Proof: These 3 claims follow after seeing that $U(x) - F$ and $x - F$ always have opposite sign, which follows from the previously established fact that $(1-x)^b = x$ has a unique root in $x \in I$. Now the curve of $y = U[U(x)]$ lies below $y = x$ for $x > F$, for example, proving the third claim.) Now any infinite sequence of points in a closed interval (namely the iterates of $U$) must have a limit point, and any limit point of this iteration (due to continuity of the iteration function and its inverse) must be a fixed point, and there is only one fixed point, so the sequence of iterates converges to $F$ starting from any $x \in I$.

Third, we observe that

$$U(F + \delta) = F + \delta/s + O(\delta^2), \quad \delta \to 0. \tag{69}$$

This easily implies that the limit in (EQ 68) exists. Finally, by applying the chain rule, we may also devise a companion iterative process (EQ 34) (EQ 35); which must converge to $\frac{\partial}{\partial q}\Psi_b^{-1}(q)$ if it converges, and it is easy to see that this iteration *does* converge. All of the arguments in this paragraph work if we start iterating from $q = F + \delta$, where $\delta$ is any *complex* number of sufficently small magnitude. Hence, we conclude that not only does $\Psi_b^{-1}(q)$ have a derivative, it in fact may be regarded (at least, in a small complex ball near $q = F$) as a complex-domained, complex-valued function having a complex derivative – hence is analytic. Since this derivative is $1/s + O(|\delta|)$ where $1/s \neq 0$, by the inverse function theorem we may now construct a function $\Psi_b(x)$, which is locally analytic near $x = 0$, which satisfies (EQ 2).

Observe that the signs of the $r_n$ in (EQ 34) alternate, which shows that $\frac{\partial}{\partial q}\Psi_b^{-1}(q) > 0$ for all $q \in I$. This rejustifies the monotonicity claim that $\Psi_b'(x) > 0$, which then consequently justifies – due to the inverse function theorem and the analyticity of $\Psi_b(x)$ – the claim that $\Psi_b^{-1}(q)$ is an analytic function of $q$ when $q$ is real and $0 < q < 1$. Observe also that $\Psi_b^{-1}(1-\delta) \to \infty$ and $\Psi_b^{-1}(\delta) \to -\infty$ as $\delta \to 0+$, all as desired.

Existence of a $\Psi_b(x)$ locally analytic at $x = 0$ is now proven, showing that all our previous claims (which depended on this assumption or some weakened version of it) were non-vacuous.

### *4.3 A second iterative process and a stronger uniqueness result*

We now will go further by analysing a second iterative process (EQ 44). It is clear that if this process converges, say in a neighborhood of $x = 0$, to a function $\Psi_b(x)$, then $\Psi_b(x)$ obeys the functional equation $\Psi_b(xs) = T_b(x)$. We now show that in fact, it does converge, even with $k = 1$, in some neighborhood of $x = 0$, to the principal solution $\Psi_b(x)$. The error in the approximation of $\Psi_b(x/s^n)$ by the Maclaurin series $f_0 + f_1 x/s^n$ is of order $O(x^2/s^{2n})$ as $x/s^n \to 0$, where $\Psi_b(x)$ is the principal (entire analytic) solution. This error is then amplified during the $n$-fold iteration of $T_b$, and assuming we take $|x|$ is sufficently small, this amplification factor is bounded by $|T'(0)|^n \cdot (1+\epsilon)^n$ for any desired $\epsilon > 0$. Notice $T'(0) = s$. Hence the total error in the $n$-step iterative process is $O(x^2(1+\epsilon)^n/|s|^n)$ which can be made arbitrarily small by taking $\epsilon$ small enough and $n$ large enough. Hence, the process does converge, to $\Psi_b(x)$, for all sufficiently small $x$ with

$k = 1$. Suppose this process converges for all $x$ with $|x| < r$, $r > 0$. Now, for any particular $x$, even if $|x| > r$, $\Psi_b(x)$ may be obtained by a finite number $j$ of $T$-iterations from a $\Psi_b(y)$ with $y = x/s^j$, $|y| < r$. Hence, the process converges for *all* real $x$.

Now, let us re-do this analysis with a little more care. The error in the approximation of $Q(x/s^n)$ – where $Q(x)$ is now *not* necessarily the principal solution, but is instead merely some solution of (EQ 2) differentiable at $x = 0$ and obeying (EQ 8) – by the Maclaurin series $f_0 + f_1 x/s^n$ is bounded by $x \cdot \iota(x/s^n)/|s|^n$ for some function $\iota(y)$ with $\lim_{y\to 0} \iota(y) = 0$. This error is then amplified during the $n$-fold iteration of $T_b$, and assuming we take $|x|$ is sufficently small, this amplification factor $A$ is bounded by $\prod_{j=1}^n T'(z_j)$ where $z_0 = f_0 + f_1 x/s^n$ and $z_j = T(z_{j-1})$ if $j \geq 1$, and $|z_j - f_0| \leq \Phi_b([1+\iota(x)]x/s^{n-j})$, $j = 0 \ldots n$. The amplification factor is thus

$$A = \prod_{j=1}^{n} T'(\iota(x/s^n)x/s^{n-j}) \leq \prod_{j=1}^{n} [f_0 + O(\iota(x/s^n)x/s^{n-j})]$$
$$\leq |s|^n \exp O(\frac{\iota(x/s^n)x}{1-s})$$

so that the total error in the approximation of $Q(x)$ by the $n$-step iterative process is bounded by

$$\leq x \cdot \iota(x/s^n) \cdot \exp O(\frac{\iota(x/s^n)x}{1-s})$$

(the constants in the $O$'s depend on $x$, but are bounded above if $|x|$ is) which tends to 0 with $s$ and $x$ fixed, $s < -1$, as $n \to \infty$. Hence the process also converges to $Q(x)$, so $Q(x) = \Psi_b(x)$!

We conclude: The principal solution $\Psi_b(x)$ of (EQ 2), (EQ 8) is the unique one which differentiable at $x = 0$.

A similar argument now will show that any solution of (EQ 2) which behaves asymptotically like $\Psi_b(0) + |x|^k \operatorname{sign}(x)$, $k > 0$, $k$ real, as $x \to 0$, is uniquely $\Psi_b(|x|^k \operatorname{sign}[x])$. Also, it shows that the only solution which behaves like $\Psi_b(0) + o(x^k)$ for all real $k > 0$, is the constant $\Psi_b(0)$, which is not admissible since it is not a CDF.

So $\Psi_k(x)$ is indeed quite unique.

### *4.4 Asymptotic behavior of $\Psi_b(x)$*

Let $\kappa(x) = 1 - \Psi_b(x)$. (EQ 2) and (EQ 3) imply that

$$\Psi_b(xs) = \kappa(x)^b \tag{70}$$

and

$$\begin{aligned}\kappa(xs) = 1 - [1-\Psi_b(x)]^b = \\ b\Psi_b(x) - \tbinom{b}{2}\Psi_b(x)^2 + \tbinom{b}{3}\Psi_b(x)^3 - \ldots\end{aligned} \tag{71}$$

where the binomial series converges for all $x$ sufficiently negative that $\Psi_b(x) < 1$. This leads to

$$\Psi_b(xs^2)^{1/b} = b\Psi_b(x) - \binom{b}{2}\Psi_b(x)^2 + \binom{b}{3}\Psi_b(x)^3 - \ldots \tag{72}$$

valid when $\Psi_b(x) < 1$.

As $x \to -\infty$, $\Psi_b(x) \to 0+$. If we let $\ell(u) = \ln[-\ln \Psi_b(\ln u)]$, then this functional equation as $u \to \infty$ is

$$\ell(u + \ln s^2) = \ell(u) + \ln b + o(1) \tag{73}$$

which certainly obeys

$$\ell(u) \sim u \ln b / \ln s^2 \tag{74}$$

as $u \to \infty$ if $\ell(u)$ is assumed to exist everywhere. The difficulty for the purpose of getting stronger asymptotic results, is that an arbitrary periodic function of period $\ln s^2$ could be added to the right hand side of (EQ 74).

Similarly, if we instead define $\ell(u) = \ln[-\ln \kappa(\ln u)]$, then we get the same results (EQ 73) (EQ 74).

Although weak, these results suffice to show via (EQ 49) that all moments exist. By differentiation of (EQ 6) one may find similar asymptotic results concerning $\Psi_b'(x)$.

Unfortunately, we were stymied in our efforts to prove more powerful asymptotic results, although we had no great difficulty conjecturing them!

For instance, suppose we assume that as $x \to -\infty$,

$$\Psi_b(x) = P \exp[-Q|x|^E + g(x)] \tag{75}$$

where $E > 0$, $g(x) = o(|x|^E)$ and $g(xs^2)/b = o(g(x))$. Then one has no difficulty in establishing from (EQ 72) that

$$P = b^{\frac{b}{1-b}}, \quad E = \frac{\log b}{\log s^2}, \quad g(x) \sim \frac{b-1}{2}\Psi_b(x), \tag{76}$$

(and note $E > 1$ as a consequence of (EQ 17)) although (unfortunately!) $Q$ *cannot* be determined by a local analysis; the only thing we know about it is that it must be positive.[1] Similarly, from

$$\kappa(us^2) = b\kappa(u)^b - \binom{b}{2}\kappa(u)^{2b} + \binom{b}{3}\kappa(u)^{3b} - \dots, \quad u \to \infty \tag{77}$$

(convergent for $\kappa(u) < 1$) we could, after making the assumption

$$\kappa(u) = P' \exp[-Q'|u|^{E'} + h(u)] \tag{78}$$

where $h(x) = o(|x|^{E'})$ and $h(xs^2)/b = o(h(x))$, obtain

$$P' = bP, \quad E' = E, \quad Q' = \frac{Q}{\sqrt{b}}, \quad h(x) \sim \frac{b-1}{2b}\kappa(u)^b. \tag{79}$$

Finally, from (EQ 49) and (EQ 79), one may obtain an asymptotic formula for the moments $\mu_k$ as $k \to \infty$:

$$\begin{aligned} \mu_k &\sim k\int_0^\infty u^{k-1}(P'e^{-Q'|u|^{E'}} + Pe^{-Q|u|^E})du \\ &\sim (Q')^{-(k-2)/E} P'E'k\Gamma(\tfrac{k-2}{E'}). \end{aligned} \tag{80}$$

This may be compared with the rigorous upper bound (EQ 52).

All this is very fine, but the author has never been able to acquire any rigorous justification of the assumptions (EQ 75) (EQ 78) in the first place, so everything from (EQ 75) to here must be regarded as conjectural.

### *4.5 Log-concavity*

The second iterative process (EQ 44) may be rephrased, it we only consider the subsequence of *even* $n$ when taking the limit as $n \to \infty$, in terms of either $V(x)$ or $W(x)$ where

$$V(x) = (1 - [1-x]^b)^b, \quad W(x) = \ln V(e^x). \tag{81}$$

[1]Presumably, $Q$ could be determined non-locally, by connection to the behavior of $\Psi_b(x)$ for small $|x|$, but this would require a depth of understanding of (EQ 2) which the author does not possess. One might hope that it would at least be possible to determine $Q$ when $b \to 1+$ via an asymptotic match to (EQ 22), utilizing (EQ 19). But this fails because the assumption $g(xs^2)/b = o(g(x))$ fails in the match-region. Nevertheless, it is somewhat informative to try to proceed with the match anyway. We observe that

$$E \to \frac{1}{2 - 2\ln 2}, \quad P \to \frac{1}{e} \quad \text{as} \quad b \to 1+$$

The $P$-limit arises directly, the $E$-limit via (EQ 14) and (EQ 18). In fact with $b = 1 + 2a$ and $a \to 0+$,

$$E = \frac{1}{1-L}[\frac{1}{2} + \frac{L}{12}\frac{3L-2}{1-L}a^2 + O(a^3)], \quad P = \frac{1}{e}[1 - a + O(a^2)].$$

We then seek a matching-point $x < 0$ and a number $Q > 0$ so that the function values match: $P\exp(-Q|x|^E) = 1/2 + x$ and the derivative does too: $|x|^{1-E}/(QE) = 1/2 + x$. Eliminating $Q$ leads to the (right hand) equation for $x$

$$|x|^E(\frac{1}{2} + x)Q = -(\frac{1}{2} + x)[1 + \ln(\frac{1}{2} + x)] = \frac{-x}{E}.$$

This match-point equation has no solution $x$ (you were warned this matching attempt was doomed to fail!) but the gap is narrowest at $x = -1/4$, which choice of matching point leads to the "conclusion" that $Q \approx 3.698$ or $Q \approx 5.875$ (depending on which right hand side is used to obtain $Q$) as $b \to 1+$.

For example, for any fixed integer $k \geq 1$, we have

$$\ln \Psi_b(x) = \lim_{n\to\infty} W^{[n]}[\ln \sum_{m=0}^{k} f_m(b) \cdot (\frac{x}{s^{2n}})^m]. \tag{82}$$

(Again, strictly speaking, we should have called $W(x)$, "$W_b(x)$," but we will regard $b$ as a fixed constant throughout this section.)

If $b > 1$ and $0 < x < 1$, then

$$V'(x) = \frac{V(x) \cdot b^2 \cdot (1-x)^{b-1}}{1 - (1-x)^b} > 0 \tag{83}$$

so $V(x)$ is a strictly increasing self-mapping of the interval $[0, 1]$. Similarly $W(x)$ is a strictly increasing self-mapping of the interval $(-\infty, 0]$. Next, if $b > 1$ and $x < 0$, then $W''(x) < 0$. To see this, write

$$-W''(x) = \frac{e^x(1-e^x)^b b^2}{(1-e^x)^2(1-(1-e^x)^b)^2} \cdot [(1-e^x)^b + be^x - 1], \tag{84}$$

and then everything is obviously positive – the term in square brackets is the only one presenting any difficulty: it is an expression of form $z^b - bz + b - 1$ where $0 < z = 1 - e^x < 1$, which is positive because it is a strictly decreasing function of $z$ and reaches 0 when $z = 1$.

Observe that the composition of two strictly increasing, strictly concave-up (respectively concave-down) functions is also strictly increasing and strictly concave-up (resp. concave-down), since $\frac{\partial^2}{\partial x^2} f(g[x]) = g'(x)^2 f''(g[x]) + f'(g[x])g''(x)$, Hence $W^{[n]}(x)$ is strictly increasing and strictly concave-down.

Now from (EQ 82) we find with $k = 1$

$$\ln \Psi_b(x) = \lim_{n\to\infty} W^{[n]}[f_1 f_0 \frac{x}{s^{2n}} + \ln f_0], \tag{85}$$

and conclude that $\log \Psi_b(x)$ is a concave-down increasing function of $x$. The inverse of a concave-down increasing function is a concave-up increasing function, so $\Psi_b^{-1}(q)$ is a concave-up increasing function of $\ln q$.

We now want to similarly prove that $\log[1 - \Psi_b(x)]$ is a concave-down decreasing function of $x$. We first rewrite (EQ 44) as an iteration to compute $1 - \Psi_b(x)$: Define $R(x) = 1 - x^b$. Then for any fixed integer $k \geq 1$,

$$1 - \Psi_b(x) = \lim_{n\to\infty} R_b^{[n]}[1 - \sum_{m=0}^{k} f_m(b) \cdot (\frac{x}{s^n})^m]. \tag{86}$$

Then we introduce new functions $J(x) = R(R[x])$ and $K(x) = \ln J(e^x)$ analagously to $V$ and $W$ above. $J(x)$ is a strictly increasing function of $x$ for $b > 1$, $0 < x < 1$, since

$$J'(x) = b^2 x^{b-1}(1 - x^b)^{b-1} > 0. \tag{87}$$

Similarly $K(x)$ is strictly increasing if $b > 1$, $x < 0$. Finally, $K(x)$ is concave-down because

$$-K''(x) = \frac{e^{bx}(1-e^{bx})^b b^3}{(1-e^{bx})^2(1-[1-e^{bx}]^b)^2}[(1-e^{bx})^b + e^{bx}b - 1] > 0. \tag{88}$$

Now with $k = 1$ and $n$ even in (EQ 86) we have

$$\ln[1 - \Psi_b(x)] = \lim_{n\to\infty} K^{[2n]} \ln[1 - f_0 - f_1 \frac{x}{s^{2n}}]. \quad (89)$$

leading to the desired conclusion that $\log[1 - \Psi_b(x)]$ is a concave-down *decreasing* (note $f_1 = 1$ and the minus sign in front of $f_1$!) function of $x$.

By inversion we find that $\Psi_b^{-1}(q)$ is a concave-up increasing function of $\ln[1 - q]$.

I conjecture that the log-odds function

$$\log \frac{\Psi_b(x)}{1 - \Psi_b(x)} \quad (90)$$

is concave-up for large $x$, and concave-down for lesser $x$, with exactly one inflection point. I have shown this when $b = 2$ and $b = 3$ [proof omitted], but not for general $b > 1$.

### 4.6 Unimodality of $\Psi_b'(x)$

By differentiation of the functional equation (EQ 2), we find

$$s\Psi_b'(xs) = -b \cdot [1 - \Psi_b(x)]^{b-1} \Psi_b'(x). \quad (91)$$

This, plus the behavior of $\Psi_b(x)$ near $x = 0$, in particular the facts that $\Psi_b'(0) = 1$ and $[1 - \Psi_b(x)]^{b-1} = -s/b + O(x)$, leads to the identity (EQ 45) and a proof it converges. Then, taking logarithms, we have

$$\ln \Psi_b'(x) = \sum_{n\geq 1} \left[ (b-1)\ln[1 - \Psi_b(\frac{x}{s^n})] + \ln \frac{-b}{s} \right]. \quad (92)$$

In view of the previously proven fact that $\log[1-\Psi_b(x)]$ is concave-down, plus the fact that sums of concave-down functions are concave-down, we conclude that $\log \Psi_b'(x)$ is concave-down.

As a trivial consequence, $\Psi_b'(x)$ is unimodal. ("Unimodal" means has exactly one maxima; maxima are called "modes" by statisticians).

The log-concavity statements we have made of course lead to inequalities concerning $\Psi_b(x)$ and $\Psi_b'(x)$. For example by considering tangent lines at $x = 0$ we derive (EQ 53).

### 4.7 Moments, inequalities

The moments $\mu_k$ of $\Psi_b'(x)$, $k = 0, 1, 2, \ldots$, are defined by

$$\mu_k = \int_{-\infty}^{\infty} x^k \Psi_b'(x) dx.$$

The other equivalent expressions in (EQ 49) arise respectively from a change of variables, integration by parts, and the use of the functional equation.

We now prove inequality (EQ 58).

$$\Psi_b'(su) = \frac{-b}{s}[1 - \Psi_b(u)]^{b-1}\Psi_b'(u) >$$
$$\frac{-b\Psi_b(0)}{[1 - \Psi_b(0)]s}\Psi_b'(u) = \Psi_b'(u), \ 0 > u.$$

The same proof works if both $>$'s are changed to $<$'s. We have used the facts that $b > 1$, $s < -1$, and that $\Psi_b(x)$ is monotonic increasing. QED.

As a consequence of this inequality, the mode of $\Psi_b'(x)$ must occur at a *positive* value of $x$. Another consequence of (EQ 58) is that all the *moments* $\mu_k$ are *positive*.

In fact, for $k$ odd, $\mu_k$ is bounded between 1 and $1-1/s^2$ times

$$\int_0^\infty x^k \Psi_b'(x) dx = k \int_0^\infty x^{k-1}[1 - \Psi_b(x)]dx. \quad (93)$$

[For $k$ even it comes as no suprise that $\mu_k > 0$, and if $k \geq 2$ is even then $\mu_k$ is bounded between 1 and $1 + 1/s^2$ times (EQ 93).] By using the upper bound (EQ 53) on the integrand in the right-hand integral of (EQ 93), we may compute an upper bound on (EQ 93) of

$$(1 - F)k \int_0^\infty x^{k-1} e^{-x/(1-F)} dx = (1 - F)^{1+k} k!, \quad k > 0. \quad (94)$$

This leads to (EQ 52).

As a consequence of unimodality we also have the inequality $\Psi_b'(-x) < \Psi_b'(x)$, $x > 0$. (Unimodality plus the fact that the mode is at positive $x$-coordinate causes $\Psi_b'(x)$ to be an increasing function of $x$ when $x < 0$ – and note $x/s > -x$ – so that $\Psi_b'(x) > \Psi_b'(x/s) > \Psi_b'(-x)$. )

Finally, bounds on the modal quantile may be deduced by writing

$$\frac{\Psi_b'(s^2 x)}{\Psi_b'(x)} = \frac{b^2}{s^2}[1 - (1 - \Psi_b(x))^b]^{b-1}(1 - \Psi_b(x))^{b-1}. \quad (95)$$

Since $\Psi_b'(x)$ is continuous and unimodal on $\mathbf{R}$, the left hand side should be 1 for exactly one real value $x$, and any such value of $x$ must occur to the left of the mode whereas $xs^2$ must lie to the right of the mode. With $\kappa = 1 - \Psi_b(x)$, we find (EQ 47). The asymptotic results (EQ 48) may then be verified by substitution.

### 4.8 More inequalities from Moebius transformations

The *Moebius transformation* is

$$M(z) = \frac{az + b}{cz + d}. \quad (96)$$

The composition of two Moebius transformations (respectively the inverse of a Moebius transformation) is another Moebius transformation whose coefficients arise from the product of the matrices (resp. the inverse of the matrix) $\binom{a\,b}{c\,d}$.

This allows one to write $N$-time iterated Moebius transformations in closed form with the aid of $n$th powers of the eigenvalues of $\binom{a\,b}{c\,d}$. The algebra is rather messy, but the result, once obtained, may readily be proved by induction on $n$ without the need for any eigenvalue computations. It is

$$M^{[n]}(z) = \frac{(F_+/F_-)[M'(F_-)]^n(z - F_-) - (z - F_+)}{(F_+/F_-)[M'(F_-)]^n(z - F_-)/F_+ \; - \; (z - F_+)/F_-} \quad (97)$$

where $F_\pm$ are the two fixed points of $M(z)$, i.e. the two roots of $M(z)=z$. Observe that there is no pole of $M(z)$ for $z$ between $F_-$ and $F_+$ if $M'(F_-)$ has the same sign as $F_+ - F_-$. ($M'(z)$ is never 0 and has the same sign as $D = ad - bc$ everywhere on the real line.)

This suggests the following procedure for approximating the $n$th iterate of any increasing self-mapping $T(z)$ of an interval $[A,B]$: Choose a Moebius transformation $M(z)$ which approximates $T(z)$ on $[A,B]$. For example, one could choose $F_\mp = A$, $F_\pm = B$, and $M'(F_-) = T'(F_-)$. Then $T^{[n]}(z) \approx M^{[n]}(z)$ as given in (EQ 97). If it can be shown that $T(z) \le M(z)$, then $T^{[n]}(z) \le M^{[n]}(z)$, and if it can be shown that $T(z) \ge M(z)$, then $T^{[n]}(z) \ge M^{[n]}(z)$.

When we adopt this procedure with the particular increasing function

$$T(z) = (1-[1-z]^b)^b = \xi + s^2(z-\xi) + \frac{(b-1)b^2\xi^2(b\xi+\xi-1)}{2(1-\xi)^4}(z-\xi)^2 + O([z-\xi]^3) \tag{98}$$

on $[0,1]$, where 0, $\xi = \Psi_b(0)$, and 1 are the fixed points of $T(z)$ in this interval and $b>1$, what happens? (Note: $b$ here has nothing to do with $b$ from the definition of $M(z)$.) We immediately find (because $T'(0)=T'(1)=0$) that it is impossible to upper bound $T(z)$ on $[\xi,1]$ by any Moebius transformation with fixed points at $\xi$ and 1, and similarly it is impossible to lower bound $T(z)$ on $[0,\xi]$ by any Moebius transformation with fixed points at 0 and $\xi$. One therefore must content oneself with upper bounds on $T(z)$ valid in $[\xi,B]$ for some $B<1$, and lower bounds valid in $[A,\xi]$ for some $A>0$.

On the other hand, choosing $F_- = \xi$, $F_+ = 0$, $M'(\xi) = s^2$ leads to a upper bound on $T(z)$ valid in the full interval $[0,\xi]$. To be precise, we claim

$$(1-[1-z]^b)^b < \frac{\xi z}{(\xi-z)s^2+z}, \quad 0<z<\xi. \tag{99}$$

We now give the (rather painful) proof. Multiply both sides by $(\xi-z)s^2+z$. Observe that equality holds when $z=0$ and $z=\xi$. The inequality is valid at all points of $(0,\xi)$ sufficiently close to an endpoint, because at $z=0$ the derivative of the right hand side is positive and of the left hand side is 0, while at $z=\xi$, the first derivatives are equal but the second derivatives are

$$\frac{(b-1)b^2\xi^2(b\xi+\xi-1)}{(1-\xi)^4} \text{ and } \frac{2(s^2-1)s^2}{\xi} \tag{100}$$

respectively. The latter quantity is larger because it may be rewritten

$$\frac{2b^2\xi(\xi^2b^2-[1-\xi]^2)}{(1-\xi)^4} \tag{101}$$

and then the difference of the two second derivatives may be written

$$\frac{[2+(b-1)\xi][(b+1)\xi-1]b^2\xi}{(1-\xi)^4}, \tag{102}$$

which is positive since $b>1$ and $(b+1)\xi>1$ from (EQ 10). Next, the derivative of the right hand side of (EQ 99) is $\xi$ and of the left hand side is

$$\left((1-s^2) + \frac{([\xi-z]s^2+z)(b-1)b(1-z)^{b-1}}{1-[1-z]^b}\right)T(z). \tag{103}$$

This function of $z$ is "loglog-concave-down," (LLCD) for $0<z<\xi$, i.e. its logarithm is a concave-down function of $\log z$. [2] Hence, the difference of the $z$-derivative of the two sides of (EQ 99) can be zero in at most 2 locations in $(0,\xi)$. A short consideration of the possibilities shows that it is topologically impossible for the number of such zeros to be 0 or 2, so it is 1, and the curves of the right hand and left hand sides of the inequality thus never cross. QED.

As a consequence of (EQ 99), we find

$$T^{[n]}(z) < \frac{\xi z}{z+(\xi-z)s^{2n}}, \quad 0<z<\xi \tag{104}$$

Letting $z=\Psi_b(x)$ in this inequality leads to

$$\Psi_b(xs^{2n}) < \frac{\xi\Psi_b(x)}{(1-s^{2n})\Psi_b(x)+s^{2n}\xi}, \quad x<0. \tag{105}$$

Upon taking a limit as $s^{2n}\to\infty$ while $xs^{2n}$ remains fixed, we obtain (EQ 56). In (EQ 105), it is by no means essential that $n$ be a positive integer. The same line of reasoning justifies (EQ 105) if $n$ is a negative integer: iteration a negative number of times means iteration of the inverse function, and the inverse Moebius transformation is also an upper bound on $T^{-1}$. Similarly, it is possible to justify (EQ 105) if $n$ is any rational number – fractional iteration of $T$ being defined by $\Psi$ itself, and fractional iteration of a Moebius function being defined by the formula (EQ 97). The justification arises from the fact that all these fractional iterates are increasing functions of $z$ which agree at the endpoints of the interval in question. Finally, by continuity we argue that (EQ 105) holds for any *real* value of $n$, and the inequality is strict except when $n=0$ or $x=0$. This leads to (EQ 54).

Similarly, choosing $F_-=\xi$, $F_+=1$, $M'(\xi)=s^2$ leads to a lower bound on $T(z)$ valid in $[\xi,1]$. To be precise, we claim

$$(1-[1-z]^b)^b > \frac{(z-\xi)s^2-(z-1)\xi}{(z-\xi)s^2-(z-1)}, \quad \xi<z<1. \tag{106}$$

The proof is similar to, but even more painful than, the proof of (EQ 99), so we omit it. As a consequence, we find

$$T^{[n]}(z) > \frac{s^{2n}(z-\xi)-(z-1)\xi}{s^{2n}(z-\xi) - (z-1)}, \quad \xi<z<1 \tag{107}$$

Letting $z=\Psi_b(x)$ in this inequality leads to

$$\Psi_b(xs^{2n}) > \frac{s^{2n}[\Psi_b(x)-\xi]-[\Psi_b(x)-1]\xi}{s^{2n}[\Psi_b(x)-\xi] - [\Psi_b(x)-1]}, \quad x>0. \tag{108}$$

[2] If one is sufficiently careful, it suffices that $T(z)$ and $1-z$ are LLCD, any positive power of an LLCD function is LLCD, the product of two LLCD functions is LLCD, and adding a positive constant to the positive LLCD function, yields an LLCD function.

Taking a limit as $s^{2n} \to \infty$ while $xs^{2n}$ remains fixed, we obtain (EQ 57), and using $y = (s^2)^n$ for general real $n$ leads to (EQ 55).

### *4.9 Behavior of $\Psi_b(x)$ as a function of $b$*

In this section, for the only time in this paper, we will consider general complex $b$, as opposed to our usual restriction that $b$ is real and $b > 1$. First, let us prove a claim which will be useful to us.

Claim: The only complex root of

$$f(z) = (z-1)^{z-1} + z^z \tag{109}$$

is at $z = 0$. (Conventions: We define $0^0 = 1$. If we slit the complex plane on the negative real axis everything is uniquely defined. There is nothing you could call a root on the slit itself, either.) Letting $z = X + iY$, $X, Y$ real, it is not hard to see that there can be no root $z \neq 0$ having either $Y = 0$, $X = 1$, or $X = 0$. By considering norms and args, any root must obey

$$([X-1]^2 + Y^2)^{X-1} = (X^2 + Y^2)^X \tag{110}$$

$$\begin{gathered}(X-1)\arctan\tfrac{Y}{X-1} + X\arctan\tfrac{Y}{X}\\ +Y\ln([X^2+Y^2][(X-1)^2+Y^2]) = 0\end{gathered} \tag{111}$$

No root of (EQ 110) can occur if $z$ is outside the union of the two unit-radius circles centered at $z = 0$ and $z = 1$. No root of (EQ 111) can occur if $|X - 1/2| > 1/2$. Therefore any nonzero root of (EQ 109) is strictly inside the rectangle with corners at $\pm i$, $1 \pm i$. Within this rectangle, however,

$$|f'(z)| < 4 + 4\ln|z| + 4\ln|z-1|. \tag{112}$$

Hence, computing all values of $f(z)$ for $z$ lying on a grid of squares of side 0.01 inside this rectangle, and verifying that $|f(z)| > 0.2$ at these points, constitutes a proof that $f(z)/z$ is nonzero everywhere. We carried out this computation. QED.

$\Psi_b(0)$ may be defined via (EQ 9) for real $b > 0$, and for complex $b$ it may be defined via analytic continuation by using the differential equation (EQ 11). This procedure in fact verifies that $\Psi_b(0)$ is analytic everywhere, except possibly where $\Psi_b(0) = 1$ or $\Psi_b(0) = 1/(1-b)$. The former case can arise only at $b = 0$ since $0^b = 1$ is required. The latter case can arise only if $f(b) = 0$, where $f(z)$ is defined in (EQ 109), which also can only occur at $b = 0$. We conclude that $\Psi_b(0)$ is analytic for all complex nonzero $b$. This fact justifies the claim that the radius of convergence of the series (EQ 14) (EQ 18) is 1/2.

Now, upon taking the partial derivative of the functional equation (EQ 2) with respect to $b$, we find

$$\frac{\partial}{\partial b}\Psi_b(xs) = \Psi_b(xs)\cdot\left(\ln[1-\Psi_b(x)] - \frac{\frac{\partial}{\partial b}\Psi_b(x)}{1-\Psi_b(x)}\right). \tag{113}$$

Also, we observe that for $|x|$ sufficiently small and $b$ bounded away from 0, $\Psi_b(x)$ is an analytic function of $b$, since $\Psi_b(0)$ is and due to term-by-term differentiation of the Maclaurin series (EQ 22) with respect to $b$ (the result is still convergent). We therefore conclude, by repeated application of the chain rule, that if $b$ is bounded away from 0, a complex derivative of $\Psi_b(x)$ is defined for all complex $x$ at which $\Psi_b(x/s^n) \neq 1$ for all integers $n \geq 1$.

In particular, $\Psi_b(x)$ is an analytic function of $b$ at all real $b > 1$ where $x$ is any fixed real.

Now, let us prove the inequality $1 < s^2 < b$, i.e. (EQ 17). [We have previously shown $-b < s < -1$ and (EQ 10).] Write

$$\frac{b}{s^2} = \frac{(1-\xi)^2}{b\xi^2} \tag{114}$$

where $\xi = \Psi_b(0)$. In view of $(1-\xi)^b = \xi$ we have $b = (\log\xi)/\log(1-\xi)$, so the desired inequality is equivalent to the well known entropy function inequality

$$\frac{(1-\xi)^2\log(1-\xi)}{\xi^2\log\xi} > 1, \quad 0 < \xi < 1/2. \tag{115}$$

Finally, it is interesting to investigate the behavior of $\Psi_b(x)$ as $b \to 1+$. In (EQ 19) (EQ 20), the claim is made that $\Psi_b(x)$ tends pointwise to a piecewise linear function with two *corners*, although, for any $b > 1$, $\Psi_b(x)$ is smooth, indeed analytic, everywhere. Similarly $\Psi_b(x)$ tends to a 2-level function with two jumps. How to prove these claims? From the Maclaurin series (EQ 22) we may observe that all coefficients $f_n$ with $n \geq 2$ tend to 0 as $b \to 1+$, while $f_0 \to 1/2$ and $f_1 = 1$, in agreement with our assertion, although hardly a proof of it. A genuine proof [but which we shall only sketch] arises from an analysis of the iterative processes (EQ 44) (EQ 45). One proves, rather straightforwardly, that if $x$ lies in any interval $[-c, c]$ with $c$, $0 < c < 1/2$, fixed, $\Psi_b(x) \to x + 1/2$, and $\Psi_b'(x) \to 1$ as $b \to 1+$. Then, due to the montonicity of $\Psi_b(x)$, the behavior for $x$ outside all such intervals, is forced.

## 5 Tables

Although the author has extensive tables, it seems fairly pointless to write them down, considering the efficiacy of the algorithms we have (EQ 32)-(EQ 43) (EQ 44)-(EQ 45) for computing $\Psi_b(x)$ and its derivatives and inverse. We will content ourselves with a few graphs and a short table of $\Psi_2(x)$.

Note with $b = 2$, $\Psi_2(0) = (3-\sqrt{5})/2$, $s = 1-\sqrt{5}$. **Table 1** gives values of $\Psi_2(x)$ and $\Psi_2'(x)$. These should be accurate except for the least significant figure.

**Table 2** gives values associated with the distributions $\Psi_b(x)$ for various $b$. In this table, $f = \Psi_b(0)$, $s = -bf/(1-f)$ is the scaling factor, *modex* is the location of the mode while *modec* is $\Psi_b(modex)$. Similarly *lix* is the location of the lower inflection point of the density function $\Psi_b'(x)$, and $lic = \Psi_b(lix)$, *uix* and *uic* arise from the upper inflection point, and the *mean* and *meanc* arise from the location of the mean. The "c" suffix in each case means the $\Psi_b()$ of the version without the suffix. Finally *stdev* is the square root of the variance. No particular accuracy is vouched for, although usually the numbers given are correct except for their least significant figure.

**Figures 1,2,3.** The graphs show the CDF $\Psi_b(x)$ and the corresponding density function $\Psi'_b(x)$ when $b = 2, 10, 50$. The locations of the lower inflection point of the density, the mode, the mean, and the upper inflection point are also shown.

**Figure 4.** The graph of $\Psi_{1.01}(x)$ is compared with the 3-line segment graph (EQ 19) of $\lim_{b\to 1+} \Psi_b(x)$.

## 6 Open problems

1. Does $\Psi_b(x)$ obey a maximum entropy (or some sort of variational) principle? (Specifically, one might conjecture that, in the original negamaxing tree problem, if the leafs were *not* independently distributed, then, the distribution of the root value would somehow be *narrower*.)
2. Find elegant and useful lower bounds on $\Psi'_b(x)$ and its moments.
3. I conjecture that $\Psi'_b(x)$ has exactly 2 inflection points (for each $b > 1$).
4. What are the asymptotics of $\Psi_b^{-1}(q)$ as $b \to \infty$ with $q$ fixed? If this question can be answered, results concerning the asymptotic behavior of the mode, the mean, and the standard deviation of $\Psi'_b(x)$ as $b \to \infty$ should then readily follow. For large, but not ridiculously large, $b$, all these things seem to behave roughly proportionally to $(\ln b)^2/b$, but for *very* large $b$, apparently one needs to enter a wierd world of iterated logarithms. Indeed, Colin L. Mallows has an argument showing that no nontrivial functions $f$ and $g$ exist so that $\Psi_b^{-1}(q) \sim f(q)g(b)$, $b \to \infty$. For these reasons, a successful analysis of this problem would be very interesting. Also, of course, our present understanding of the asymptotics of $\Psi_b(x)$ as $|x| \to \infty$ with $b$ fixed, is rather pitiful.
5. The *best move* in a tree is the child of the root with the minimal negamax value. One definition of the *conspiracy number* of a tree is the minimum number of leafs whose value must be changed in order to change the best move. Conjecture: the conspiracy number $C$ of a depth-$h$ tree with branching factor $b$, assuming the leafs are drawn independently from any smooth probability density (or equivalently, are ordered according to a random permutation) almost surely, as $h \to \infty$ with $b \geq 2$ fixed, obeys $(\log_b C)/h \geq a > 0$ where $a$ is some absolute constant. (Probably $a = .499$ suffices.)

## 7 Notes and Acknowledgements

Judea Pearl [4] [5] was the first to observe, and to use, many of the facts in this note. In fact, he used properties of $\Psi_b(x)$ to find that the "alpha-beta" algorithm for evaluating the negamax value of a height-$h$ branching-$b$ tree, whose leaf-values were ordered according to a random permutation, almost surely took time $T$ where

$$T^{1/h} \sim \frac{b}{|s|} = \frac{1 - \Psi_b(0)}{\Psi_b(0)} \quad \text{as } h \to \infty \tag{116}$$

and that (in this probabilitistic model) the right hand side was minimum possible for any negamaxing algorithm.

However, Pearl's treatment was neither rigorous, complete, nor correct.[3]

Theorems concerning existence, uniqueness, and global-attractor properties for solutions of a wide class of iterative functional equations related to (EQ 1), may be found (plus a great deal more) in [3].

The techniques of the present paper may also be used to analyse certain other processes on regular trees. One process of interest, also alluded to by Pearl (and attributed by him to D. Cantor, although the same sort of idea was once studied in a paper by John Tukey [7]) is one in which the value of an interior node of the tree is the *median* of its $b$ children, where $b > 1$ is an odd integer. (This scheme may be used to find approximate medians in an "on-line" manner.) In this case, the CDF recurrence (EQ 1) must be replaced by

$$\Phi_h(x) = P_b[\Phi_{h-1}(x)] \tag{117}$$

where the polynomial $P_b[y]$ is defined by

$$P_b[y] = \frac{\int_0^y x^{(b-1)/2}(1-x)^{(b-1)/2} dx}{\int_0^1 x^{(b-1)/2}(1-x)^{(b-1)/2} dx}. \tag{118}$$

A unique analytic self-similar fixed point CDF $\Theta(x)$, analagous to $\Psi_b(x)$, again exists, obeying

$$\Theta_b(sx) = P_b[\Theta_b(x)], \tag{119}$$

where $\Theta'_b(0) = 1$, $\Theta_b(0) = \frac{1}{2}$, $\Theta_b(x) + \Theta_b(-x) = 1$,

$$\begin{gathered} \Theta'_b(x) > 0, \\ \lim_{x\to\infty} \Theta_b(-x) = 0, \quad \lim_{x\to\infty} \Theta_b(x) = 1. \end{gathered} \tag{120}$$

and

$$s = P'_b[\frac{1}{2}] = \frac{\binom{b-1}{(b-1)/2} b}{2^{b-1}} > 1. \tag{121}$$

The closed form solution of (EQ 117) (corresponding to (EQ 4)) is then

$$\Phi_h(u) = \Theta_b(s^h \Theta_b^{-1}[\Phi_0(u)]). \tag{122}$$

In particular from (EQ 121) with $b = 3$ we find $s = 3/2$, (this figure was given incorrectly by Pearl [4]), with $b = 5$, $s = 15/8$, and with $b$ large, $s \sim \sqrt{2b/\pi}$. This

[3] Of his 7 claims leading up to his theorem 2, Pearl's "proofs" of claims 4,5, and 7 are unsalvagable. It is conceivable, though unlikely, that those claims may still be true, but certainly neither Pearl nor I know how to prove them. The author's realization of this was one fact which inspired him to write the present paper. Fortunately, these claims were apparently not among the properties Pearl needed for his alpha-beta analysis, thus the important result (EQ 116) is still valid.

scaling factor $|s|$ is considerably larger than the corresponding scaling factor in negamaxing with the same $b$; cf. (EQ 15) (EQ 16) (EQ 17), so that approximate median finding schemes based on median-of-children rules should (not suprisingly) be *better* than approximate median finding schemes based on negamaxing, from both the standpoints of accuracy and of simplicity.[4]

## 8 References

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[4] This is directly contrary to the conclusion apparently arrived at by Pearl [4].