Fundamental physical limits on information storage

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1 Introduction

This paper[1] argues that

Claim 1 [2] *The maximum number I of bits of information that can be stored in a box of volume V and total gravitating mass-energy M is (only claimed to be valid asymptotically for I/V large)*

$$\frac{I}{V} \lesssim \kappa_2 \, (\frac{M}{V})^{3/4}, \quad \text{where} \quad (\frac{2}{q})^{1/4}\kappa_2 = \frac{16}{3\ln 2}\frac{(\sigma_{SB})^{1/4}c^{5/4}}{k_B} \approx 3.392 \times 10^{32} \, \frac{\text{bit}}{\text{Kg}^{3/4}\,\text{meter}^{3/4}}, \quad (1)$$

where q is the number of "types" of particles in the universe and $\lesssim$ means "less than something asymptotic to."

Thus in a universe in which the only allowed particles were photons – available in 2 polarizations – we would have $q = 2$ causing κ_2 to equal the numerical value given in (EQ 1). (Our results are pretty insensitive to q since they depend on $(q/2)^{1/4}$.) This would be asymptotically tight since a blackbody radiation bath of total mass-energy M filling the box has this entropy [31].

Here is some physical reasoning to support this bound, in which item 3 is nonrigorous.

1. If we are going to place particles in the box, bosons will give us more information storing capacity than fermions, all other things (e.g. particle masses) being equal, since we won't be constrained by Pauli's exclusion principle.

2. If we are going to use an "ideal gas" of noninteracting bosons, massless bosons are better than massive ones because they have shorter De Broglie wavelength λ at the same energy E according to
$$\lambda = \frac{h}{p}, \quad E = \sqrt{m_0^2c^4 + p^2c^2} \qquad (2)$$
allowing us to cram in more waves, and hence more quantum mechanical states of the system, and hence more bits of information.

3. Finally, the number of states of a quantum system is roughly the volume of "phase space" accessible within the energy and volume constraints. Any interaction among particles tends to decrease the accessible volume of phase space, so we make the *ideal gas conjecture* that the entropy of a set of N particles with given masses, fixed total kinetic energy, and constrained within a region of fixed volume, cannot be made much larger than the entropy arising from the assumption that the particles were noninteracting[3].

This leads us to the conclusion that the maximum amount of information arises if the box is filled with a gas of noninteracting massless bosons (e.g. photons) in their maximum entropy state (i.e., a blackbody radiation bath). Then (EQ 1) follows immediately from the known formula [31] for this entropy (converted to a statement about bits by using the fact that thermodynamic entropy is $k_B \ln 2$ times information theoretic entropy), modified to allow q kinds of particles rather than just 2.

We are going to justify this heuristic reasoning by giving 2 proofs of the asymptotic validity of the ideal gas conjecture assuming either Schrödinger's equation, or variant equations designed to be caricatures of relativistic quantum mechanics, and assuming the particles have square-integrable (over the box) 2-body potentials (for example, Coulomb potentials in 3D will do[4]).

[1] Physical constants we use: Stefan-Boltzmann's σ_{SB}, Planck's h, $\hbar$, Boltzmann's k_B, and the speed of light c.

[2] Immediate consequences include bounds on information flux in terms of mass-energy flux (this will be discussed in a companion paper), and mathematical justification of the physical idea that the physical process which gets rid of heat fastest, in the asymptotic limit of high heat flux, is blackbody radiation; convection and conduction eventually do not help. Additional arguments about details we cannot discuss here, such as what to do when the box is very flat in shape, comparative discussion of different elementary particles, and why "readout" of the memory is possible, arguably even in times comparable to the lightspeed delay, are in a larger 1992 NECI tech. report on my web page [47].

[3] This conjecture also seems to make sense experimentally, since everything seems to behave like an ideal gas at its highest entropy state (i.e. when heated sufficiently).

[4] Or potentials with -1.499 power law singularities, in 3D.

1.1 Sketch of rest of paper

We'll discuss previous work in §2. Readers solely interested in our result and proof can skip §2 entirely, since only the numerical comparison in §2.1 has any relevance. The strategy of the two proofs in §3 are roughly as follows. We want to show that the multiparticle Schrödinger equation with *any* potential function (from a wide class) asymptotically at high energies has the same spectrum as the much simpler "wave equation" obeyed by noninterating particles. The easier way to do this is to use a result of Kato (our lemma 7) on "relative infinitesimal boundedness.") The harder (but more rewarding) way is to view the potential energy operator as a small perturbation of the kinetic energy operator. If these operators were actually finite dimensional matrices, then the famous "Hoffman-Wielnadt theorem" would enable bounds on the spectral perturbation to be deduced. We extend this theorem to hold for operators, under certain conditions, as well.

2 Previous work on entropy bounds

2.1 Bekenstein's black hole bound

Bekenstein [3] argued that thermodynamic statements about black holes lead indirectly to a remarkable upper bound on the thermodynamic entropy S of *any* physical system of gravitating mass M enclosed by a surface of area A. Expressed in terms of information-theoretic entropy, Bekenstein's bound is

Claim 2 (Bekenstein 1981) *If A is the area of a surface enclosing a computer memory (of volume V and gravitating mass M) storing I bits of information, then*

$$I < \kappa_6 M\sqrt{A}, \quad \text{where} \quad \kappa_6 = \frac{\sqrt{\pi}c}{\hbar \ln 2} \approx 7.267 \times 10^{42} \ \frac{\text{bit}}{\text{Kg m}}. \tag{3}$$

This bound is tight in the sense that the right hand side is precisely equal to the "entropy" of an uncharged nonrotating black hole[5] with mass M and surface area (of the event horizon) A. Also from [49] it appears that Bekenstein's bound is only off by a small constant factor for certain "thermal geons," i.e. balls of blackbody radiation in equilibrium with their own gravity.

Bekenstein's bound has the virtues that it does not need to make any assumption about q (the number of particle types) and remains valid even in situations where the mass-density is so large that the metric of space (and A) is distorted by general relativistic effects – whereas our bound (EQ 1) neglects and ignores general relativity.

But on the other hand: there are still some loose ends in Bekenstein's argument underlying this bound, which have not yet been tied up [47]. Also the present paper would seem to indicate that Bekenstein's bound is weaker than it "should" be, since with physically sensible parameter values, our bound is usually a lot stronger than Bekenstein's. For example, a 1cm cube weighing 10^6Kg (presumably at this high a density, we are safely[6] in asymptopia!) could contain $\le 3.4 \times 10^{35}$ bits, according to (EQ 1). Bekenstein's weaker bound (EQ 3) yields 1.7×10^{47}.

2.2 Spectral bounds

A different approach to bounding information storage capacity is to find the count C of the number of eigenstates of a quantum system with energy below some bound E, and then proclaim that $\log_2 C$ bounds the maximum number of bits we can store.

Let $e_1 \le e_2 \le \cdots$ be the eigenvalues (physically e_j are the "spectral energies," although for the present discussion we are getting rid of all the physical constants and units) of a Schrödinger operator in d-dimensional space

$$H = U(\vec{x}) - \nabla^2. \tag{5}$$

(U is the "potential energy function.")

[35] [9] [45] [7] independently showed that an inequality of the form

$$\sum_{\substack{j \\ e_j \le E}} |e_j|^\gamma \le L_{\gamma,d} \int_{\vec{x}:U(\vec{x})\le E} |E - U(\vec{x})|^{\gamma+d/2} \mathrm{d}^d\vec{x} \tag{6}$$

holds, for suitable finite positive constants[8] $L_{\gamma,d}$, if $\gamma \ge 0$ and $\gamma > 1 - d/2$. An old result of this sort in 1D (which

[5] From Hawking's temperature formula [18], the information theoretic entropy I of an uncharged nonrotating hole (the rightmost expression is valid even for charged rotating holes where A is the area of the event horizon) is

$$I = \frac{4\pi G}{\hbar c \ln 2} M^2 = \frac{c^3}{4\hbar G \ln 2} A \tag{4}$$

and (EQ 3) arises by taking the square root of the product of these 2 expressions.

[6] Roughly speaking, (EQ 1) should be valid when densities are high enough that the typical kinetic energy per particle far exceeds the binding energy of 2-particle potentials. At lower energies, it is not going to be valid, because asymptotically negligible terms won't be. Nevertheless, even at "low" everyday densities such as $M = 1$Kg, $V = 10^{-3}\text{m}^3$ (a liter of water) we get reasonable answers (6.0×10^{31} bits) not much larger than the number (1.3×10^{29} bits) you might expect by considering the fact that a liter of water contains 9.4×10^{26} particles (electrons, protons, and neutrons) and assuming each particle represents $3\log_2(10\text{cm}/1\text{fm}) \approx 140$ bits of positional information. This is rather nice considering that the derivation of (EQ 1) knew nothing whatever about electrons, protons, and neutrons.

[7] But there are errors in [9], and proofs are omitted in [45]. For further discussion of results like this: [36] [46] [13] [38] [42].

[8] Good bounds on these constants are as follows. First, $L_{\gamma,d} \le L_{0,d} \frac{\gamma!\,(d/2)!}{(\gamma+d/2)!}$ so we only discuss $L_{0,d}$. The "classical bound" is

$$L_{0,d} \ge \frac{\CIRCLE_d}{(2\pi)^d}. \tag{7}$$

Cwikel [9] claims the upper bound

$$L_{0,d} \le \frac{d^{d+1}}{64(d-2)^{d-1}} (\frac{2}{\pi})^d \CIRCLE_d, \tag{8}$$

which is $O(4^d d^2)$ times larger than the lower bound. In particular: $0.0168 < 1/(6\pi^2) \le L_{0,3} \le 27/(2\pi^2) < 1.37$. Lieb's bound, also in some ways has not been improved upon) is [17]. Thus if $d \geq 3$ we may take $\gamma = 0$ to get an entropy bound.

In the wave equation (free particle) case $U \equiv 0$ on a fixed Riemannian d-manifold Ω, H.Weyl in 1911 [51] had shown that as $E \to \infty$

$$\#\{e_j \leq E\} \sim (2\pi)^{-d} \bullet_d E^{d/2} \mathrm{Vol}_d(\Omega), \quad \text{i.e.}$$
$$e_k \sim 4\pi^2 [\frac{k}{\bullet_d \mathrm{Vol}_d(\Omega)}]^{2/d} \qquad (11)$$

where $\bullet_d = \pi^{d/2}/(d/2)!$ is the volume of an d-dimensional ball of unit radius. Simplifications of this and related results are discussed in [23] [8] [6] [43] [9] [34] [46]. For a d-box (Cartesian product of n finite real intervals) exact expressions for the eigenmodes are known; in the case of a unit d-cube the eigenvalues e are $\pi^d \sum_{j=1}^d n_j^2$ for d-tuples of positive integers $\vec{n}$.

Lieb's bound may be validly applied for multiparticle systems by simply viewing N particles as 1 particle in a space with N times as many dimensions (some of them rescaled, if the particles have different masses).

Unfortunately, despite this, Lieb's bound (EQ 7) is usually *useless* except in 1-particle scenarios for small γ. That is because, e.g., if the particles interact Coulombically, the right hand side of (EQ 6) contains divergent integrals if $\gamma + d/2$ is large enough, so the upper bound is ∞.

2.3 Conjectural improvement

What is the physical intuition behind (EQ 6)? Why the power $\gamma + d/2$? In the semiclassical approximate picture of quantum mechanics, a mass-m particle with kinetic energy K has wavelength $h/\sqrt{2mK}$; in other words (since we are ignoring physical constants such as Planck's h) proportional to $1/\sqrt{E-V}$ for a particle of total energy E. So, noticing from (EQ 11) that the number of modes whose characteristic wavelength is of order $\geq \lambda$ grows proportionally to λ^{-d} per unit d-volume, we would expect the number of modes of energy $\leq E$ to grow proportionally to $\int (E-V)^{d/2} \mathrm{d}^d\vec{x}$. (This heuristic argument is easily generalized to allow $\gamma > 0$, too.) We only integrate over the classically allowed region $\{\vec{x} : U(\vec{x}) \leq E\}$. This explains the intuition behind the bound in the "semiclassical regime" $\hbar \to 0$; and the hope [46] is that even when we are not in this regime, the bound (perhaps with a larger constant) still will remain valid[9].

But, suppose we have n particles in (say) $\mathbf{R}^3$ interacting (say) Coulombically. When two attracting particles are close but the others are well separated, the potential energy gets very negative. So the kinetic energy of one or two particles gets large, but the others are not affected. If we foolishly regard the N particles as 1 superparticle in $3N$ dimensional space, then we suffer by raising the wavenumber to the power $3N$. Really, we should only raise it to the power 3 or 6 because only 1 or 2 of the N particles have small wavelength. In other words, the superparticle has small wavelength (highly oscillatory wavefunction) *only in 3 or 6 directions* corresponding to the coordinates of those 2 particles, and if you walk in the remaining $3N-3$ or $3N-6$ directions, the wavefunction is slowly varying. So we should *not* be taking the power N. This explains why Lieb's inequality is weak in multiparticle scenarios, and suggests conjecture 3:

Consider the N-particle Schrödinger operator in d-space

$$H = \sum_{j=1}^{N-1} U_j(\vec{x}) - \nabla_j^2. \qquad (12)$$

Here $\vec{x}$ is an $(N-1)d$-dimensional vector of all but[10] one of the particle coordinates, U_j is the potential felt by particle j, and ∇_j^2 is the Laplacian for the d coordinates of particle j. Then

Conjecture 3 *There exist suitable finite positive constants $W_{\gamma,d}$, $\gamma \geq 0$ and $\gamma > 1 - d/2$, so that*

$$\sum_{\substack{j \\ e_j \leq E}} |e_j|^\gamma \leq W_{\gamma,d} \max_{\substack{E_1, E_2, \ldots E_{N-1}, \\ E_1+E_2+\cdots+E_{N-1}=E}}$$
$$\int_{\vec{x}:U(\vec{x})\leq E} \prod_{j=1}^{N-1} |E_j - U_j(\vec{x})|^{\gamma+d/2} \mathrm{d}^{(N-1)d}\vec{x}. \qquad (13)$$

3 Our result

Conjecture 3 with $\gamma = 0$ would have sufficed to prove claim 1 for Coulombic particles; but after failing to prove it, I adopted a different and successful approach based on the Hoffman-Wielandt theorem.

3.1 Matrix/operator terminology

We try to use the same words for matrices and (linear) operators throughout. A matrix A is "Hermitian" if $A^H = A$, where A^H denotes the complex-conjugate

discussed in [46] [13] [38] [24] states that

$$L_{0,d} \leq (2\pi)^{-d/2} \int_0^s s^{-1-d/2} f(s) \mathrm{d}s \qquad (9)$$

for any concave-∪ function $f(s)$ such that $\int_0^\infty e^{-y} y^{-1} f(2y) \mathrm{d}y = 1$. In particular one may use $f(y) = \max\{0, (y-b)/a\}$ for any $a, b > 0$ with $2e^{-b} - b\int_b^\infty e^{-y} y^{-1} \mathrm{d}y = a$. This leads to

$$L_{0,d} \leq \frac{4e^b b^{1-d/2}}{(2\pi)^{d/2}(2d-4+[2-d]be^b \mathrm{Ei}_1(b))d}, \quad \mathrm{Ei}_1(x) = \int_1^\infty e^{-xt} t^{-1} \mathrm{d}t \qquad (10)$$

for any $b > 0$. With $b = .38325$ we get $L_{0,3} \leq 0.1262680095$. For d large, with the approximately optimal choice $b = (d-2)/2$, Lieb's upper bound (EQ 10) grows roughly like 1.43^d times the classical lower bound.

[9] One might conjecture that (EQ 6) should be asymptotically exact in the limit when the potential is multiplied by a constant scaling factor and this factor is made large, because then semiclassical approximation ought to be exact. But this claim of Cwikel [9] (and [46] theorem 10.7) is assuredly incorrect for potentials for which the integral diverges.

[10] We let the origin of our coordinate system be the center of mass of the particles; the Nth particle's coordinates are then deducible from the other $N-1$ particles's coordinates.

transpose. These have real eigenvalues. A is "unitary" if $A^H = A^{-1}$. The "Frobenius norm" (also called the "Hilbert-Schmidt norm" [44] [10] [1]) $\|A\|_F$ of a matrix is

$$\|A\|_F = \sqrt{\sum_j \sum_k |A_{jk}|^2} = \sqrt{\mathrm{trace}(A^H A)}. \tag{14}$$

A set S is "compact" if every infinite sequence of elements from S contains a convergent subsequence such that the limit of this subsequence lies in S. (A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded.) An operator T is "compact" (this notion is disjoint from the notion of compactness for *sets*) if $\Phi_n = T\Psi_n$ contains a convergent subsequence for every L_2-norm-bounded sequence $\Psi_1, \Psi_2, \ldots$. An equivalent definition of compactness is that the ball $|\Psi| \le 1$ is mapped by T to a compact set. An operator T is "bounded" if its norm $\|T\| = \sup_{\Psi, |\Psi|=1} |T\Psi|$ is bounded. Compact operators are automatically bounded. An operator is "bounded below" if $\inf_{\Psi, |\Psi|=1} |T\Psi|$ is bounded below. The "domain" $\mathrm{dom}(T)$ of an operator T is the set of Ψ in the underlying Hilbert space such that $T\Psi$ is defined.

The "resolvent" of an operator T is $(\lambda I - T)^{-1}$, where λ is a complex parameter. The "spectrum" of T is the complement of the set of λ such that the resolvent is bounded. The "essential spectrum" is the set of accumulation points (limits of infinite subsets) of the spectrum. An operator T is "self-adjoint" if it is Hermitian and $\mathrm{dom}(T) = \mathrm{dom}(T^H)$. Self-Adjoint operators have spectra forming a closed set in $\mathbf{R}$ ([1], theorem 4 section 43 vol. 1). For operators which are not finite dimensional matrices, self-adjointness is *not* an automatic consequence of Hermitian symmetry, and can be tricky to establish.

3.2 "Normal" matrices

A matrix A is (this is a very bad name, but by now conventional usage) "normal" if $A^H A = AA^H$. It may be shown that A is normal iff it has a factorization $A = QDQ^{-1}$, Q unitary and D (complex) diagonal. Also, A is normal iff the sum of the absolute squares of its eigenvalues is equal to the the sum of the absolute squares of its elements; for nonnormal matrices, the latter sum is greater. The reason (if there can be said to be one) for the name "normal" is that Hermitian (and hence real symmetric), orthogonal, unitary, and antiHermitian matrices are automatically normal.

3.3 Strategy

We would like to regard the Schrödinger operator $U - \nabla^2$ as a *perturbation* of the better understood operator $-\nabla^2$, and we would like to know how this perturbation affects eigenvalues. Quantum mechanical "first order perturbation theory" claims that an eigenvalue λ corresponding to an eigenfunction ψ of some operator A, will, when A is perturbed to $A + \delta$, move by approximately

$$\int \psi(\vec{x})^* \, \delta \, \psi(\vec{x}) \, \mathrm{d}^n \vec{x}. \tag{15}$$

We need a rigorous bound on eigenvalue motion, valid even for possibly large perturbations of the operator – as opposed to (EQ 15), which is an asymptotic statement valid only for suitably infinitesimal perturbations.

3.4 The Hoffman-Wielandt theorem

The Hoffman-Wielandt theorem [21] (a mainstay of numerical linear algebraicists trying to make rigorous "backward error" analyses [52]) was designed to solve this problem for $m \times m$ Hermitian matrices. Although these aren't as general as Hermitian linear operators, it's a step in the right direction. We divide both sides of the bound in this theorem by m in order to write it as a bound on the mean square eigenvalue perturbation:

Theorem 4 (Hoffman & Wielandt 1953) *If A and B are $m \times m$ Hermitian matrices with respective (real) eigenvalues α_j, β_j, sorted in nondecreasing order, then*

$$\frac{1}{m}\sum_{j=1}^{m} |\beta_j - \alpha_j|^2 \le \frac{1}{m}\|A - B\|_F^2 = \frac{1}{m}\sum_{j=1}^{m} |\gamma_j|^2 \tag{16}$$

where γ_j are the eigenvalues of $A - B$ in non-decreasing order, and this bound is tight.

Remarks:
(i) Hoffman & Wielandt actually showed (stronger) that this holds if A and B are $m \times m$ *normal* matrices, for *at least one* ordering of the eigenvalues α_j, β_j.
Simple Proof of this stronger theorem [16]. Without loss of generality let A be diagonal, and let $UBU^H = D$ be diagonal for some unitary U. Then

$$\|A - B\|_F^2 = \|A - U^H DU\|_F^2 = \|UA - DU\|_F^2 \tag{17}$$

$$= \sum_{j,k} |U_{jk}|^2 |A_{jj} - D_{kk}|^2 = \sum_{j,k} |U_{jk}|^2 |\alpha_j - \beta_k|^2 \tag{18}$$

$$= \sum_{\sigma \in S_m} \sum_{k=1}^{m} w_\sigma |\alpha_j - \beta_{\sigma(k)}|^2 \ge \min_{\sigma \in S_m} \sum_{k=1}^{m} |\alpha_k - \beta_{\sigma(k)}|^2. \tag{19}$$

We've used the fact that the matrix with elements $|U_{jk}|^2$, is doubly stochastic (has non-negative elements, and all row sums and column sums are 1) and hence by the Birkhoff-König theorem is a convex combination (with coefficients we've called "w_σ") of permutation matrices[11]. □

(ii) For related results see [2] [19] [4] [30] [50] [48].

(iii) Most interestingly for our purposes, there are generalizations of this theorem to allow the "$m \times m$ matrices" to instead be *linear operators* in a Hilbert space, possibly an infinite dimensional one [30] [4] [12]. But none of these generalizations seem to cover the case we want – they all seem to require that $\|A - B\|_F$ be finite.

What we will need to know is that

[11] σ is a permutation selected from the group S_m of permutations of m items.

Theorem 5 (Our version of Hoffman-Wielandt) *Theorem 4 holds in the limit of infinite Hermitian matrices ($m \to \infty$) provided both sides are defined. More precisely, let A and B be Hermitian linear operators with finite least eigenvalue, countable spectra without accumulation points (except possibly for the ground state), and complete orthonormal sets of eigenfunctions. This is in the Hilbert space [20] [44] [10] [1] of L^2 functions with compact support $\Omega \subset \mathbf{R}^n$ (say for Ω with sufficiently smooth boundary*[12] *and Dirichlet boundary conditions), and assuming $\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} |\gamma_j|^2$ exists where γ_j are the eigenvalues of (or the Euclidean row lengths of an $m \times m$ section of) $A - B$ with the $|\gamma_j|^2$ ordered in nondecreasing order.*

Proof. What are the obstacles that prevent the "simple proof" given before from generalizing immediately to handle the present problem? First, A and B need not have countable spectra, or could have multiple accumulation points in their spectra, and could have both unboundedly negative and positive eigenvalues, making the meaning of matching up the "nth" eigenvalue of A with the "nth" eigenvalue of B unclear. This is OK since we've assumed A and B both have "countable spectra" without accumulation points and "finite least eigenvalue." Second, although A and B are diagonalizable automatically if they are finite normal matrices, this is not necessarily the case for infinite matrices – but this is OK since we have assumed "complete orthonormal sets of eigenfunctions." Third, the justification of the final line (EQ 19) required the Birkhoff-König theorem, which again applies only for finite doubly stochastic matrices. The way around that obstacle is to use a modified version of theorem 1 of [12] in its stead.

Such a modified version may be shown by modifying the proof in [12]. Essentially, in the proof of [12]'s theorem 1, we need in (2.1) and (2.2) to divide both sides by m, make the sum go up to m, and take the limit as $m \to \infty$. Similarly adjust the final unnumbered equation in that proof. In (2.{5,6,7,8}) the right hand side is multiplied by $2n$ and K is made a function of n which tends to a constant in the limit. Their theorem 2 then follows. (More generalizations of [12] should be possible, e.g. considering raising m to powers other than -1.) □

Remarks:
(i) The Riesz-Schauder theorem ([44] vol. 1 Theorem VI.15) says that compact operators have countable spectra with finite eigenvalue multiplicities, and without limit points (except perhaps at 0).

(ii) The Hilbert-Schmidt theorem ([44] vol. 1 Theorem VI.16) says that "self-adjoint" compact operators have a complete orthonormal set of eigenfunctions. For demonstration of the fact that the Hamiltonians based on Coulombic, or more generally $L^2(\mathbf{R}^3) + L^\infty$ interparticle potentials, have a complete orthonormal set of eigenfunctions, see [25] [27][14]. The converse fact that the radially symmetric potential $-K/r^2$, for large enough constant K, is *not* self-adjoint (and indeed the particle "falls into the singularity" in finite time) and hence unsuitable for quantum mechanics is mentioned both there and in [41] [39].

(iii) For the fact that the Green's function (also called the "resolvent") of the Laplacian operator (in compact 3D boxes Ω with sufficiently smooth boundary with Dirichlet boundary conditions) is a compact self-adjoint operator, see sections 7.2 and 7.8 of [40]. (The Laplacian itself is of course unbounded and hence noncompact.) Hence by (i), the Laplacian operator on Ω has a countable spectrum with no accumulation points, as we anyway already knew from Weyl's asymptotic (EQ 11). The argumentation in [44] vol. 4 theorem XIII.14 and corrollary 2 and example 6 (but altered by considering potential functions defined only on a *compact* region Ω in $\mathbf{R}^3$, rather than all of $\mathbf{R}^3$) shows that any potential function $U(\vec{x})$ in $L^2(\mathbf{R}^3) + L^\infty$ is[13] "relatively compact" compared to the Laplacian operator, and therefore the essential spectrum of the Laplacian and Schrödinger operators over Ω must be identical, i.e. the empty set, in other words proving that the Schrödinger operator arising from such a potential function also must have a countable spectrum with no accumulation points.

3.4.1 How to use our Hoffman-Wielandt type theorem

Then, in theorem 5 the negated Laplacian operator, i.e. kinetic energy operator, on some bounded region $\Omega \subset \mathbf{R}^n$ applied to L^2 functions with Dirichlet boundary conditions (0 on $\partial\Omega$) will play the role of "A." Let B be the Schrödinger operator $U(\vec{x}) - \nabla^2$ on the same region; $B - A$ is the potential function $U(\vec{x})$.

Clearly A and B are Hermitian. Assume U is a Coulombic multiparticle potential. Then A and B are self-adjoint and hence have complete orthonormal sets of eigenfunctions [25] [27][14]. They also have finite least eigenvalue: this, assuming Coulomb potentials, or indeed much more general interparticle potentials, follows from Lieb's and others's proofs of the "stability of matter" [32] [33], and also from lemma 1 of [26] combined with lemma 7 below. The absence of accumulation points in the spectrum of A is immediate from Weyl's formula or from results mentioned in [43]. Remarks (i), (ii), and (iii) after theorem 5 show the countability of the spectra and the completeness of the eigenfunctions of these operators, etc., as needed. In other words, we are allowed to apply theorem 5.

It is best to keep the following specific example in mind. Imagine that Ω is an n-box, and we write everything in the eigenbasis of the Laplacian, i.e. the or-

[12] For example, it will suffice if Ω is convex, bounded, has an interior, and satisfies the "Poincare ball property" that Ω may be written as the union of a denumerable set of balls.

[13] As [44] also show, we may allow dimensions other than 3 if the potential obeys the same conditions as in footnote 17. Also, instead of $L^2(\mathbf{R}^3) + L^\infty$, the considerably wider class of "Rollnik" potentials, is also allowable [44].

[14] For an extension to all dimensions ≥ 1 (rather than just 3D) under the same conditions on $U(\vec{x})$ as in footnote 17, see [5]. For allowing smooth constant magnetic fields, see [22].

thonormalized Fourier modes $F_j(\vec{x})$ of the box. In that case, A is diagonal and the jkth entry of B is

$$B_{jk} = \int_\Omega F_j(\vec{x})\, U(\vec{x})\, F_k(\vec{x})^*\, \mathrm{d}^n\vec{x}. \tag{20}$$

(If we want to stay general and allow Ω to be a non-box, then make F_j be the eigenfunctions of the Laplacian, with Dirichlet boundary conditions. The fact that the F_j always form a "complete" orthonormal basis follows from Hilbert's "spectral theorem [7]" for "completely continuous" operators.) Plainly A and B are Hermitian.

3.5 Side remark

The following reasoning, designed to get bounds on $|B_{jk}|$, ultimately did not seem useful for my proof. However, it can be used to establish lemma 7 and the consequent alternate proof in §3.9. We have the obvious bound

$$|B_{jk}| \le \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega |U(\vec{x})| \mathrm{d}^n\vec{x} \equiv \overline{V}. \tag{21}$$

But for specific potentials, we could say more. First, if the potential has mean value $\overline{V}$ on Ω, then $B_{jj} = \overline{V}$. Second, we could use the closed form expression [53][15] for the Fourier transform of a 3D screened Coulomb potential

$$U(\vec{x}) = \frac{q^2}{r} e^{-\theta r}, \quad r = |\vec{x}|, \tag{22}$$

namely

$$F(\vec{k}) = \frac{4\pi q^2}{\alpha^2 + |\vec{k}|^2} \tag{23}$$

to see that if $j \ne k$ and our potential is that of N Coulombic particles (each with charge $\le q$ in magnitude) in a 3D box, then $|B_{jk}|$ falls off as the inverse square of wavenumber.

3.6 Back to main argument

It then follows from theorem 5 that the eigenvalues β_j of the Schrödinger operator $U(\vec{x}) - \nabla^2$ on Ω do not differ very much from the eigenvalues α_j of $-\nabla^2$ on Ω, specifically

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} |\beta_j - \alpha_j|^2 \le \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m}\sum_{k=1}^{m} |B_{jk}|^2 \le \overline{V^2}. \tag{24}$$

(We have used Parseval's identity to bound the sum of the absolute squares of the elements in each row of B by $\overline{V^2} \equiv \mathrm{Vol}(\Omega)^{-1} \int_\Omega U(\vec{x})^2 \mathrm{d}^n\vec{x}$.) Thus, the RMS perturbation to the eigenvalues will be bounded by the integral of the square of the potential – a constant if that integral converges. It is easy to see that such perturbations are incapable of affecting the asymptotic (EQ 11), so all interesting asymptotics of the behavior of the eigenvalues are the *same* for a flat-bottomed box and any other potential in a box arising from $L^2(\mathbf{R}^3)$ interparticle potentials. Hence

Theorem 6 (Main result) *Suppose N particles (from a fixed finite set of permissible particle types) in a 3D box Ω interact according to a fixed finite set of 2-body potentials, each having bounded mean square and mean over Ω. Then assuming Schrödinger's equation holds, and assuming its ground state has energy bounded below, their entropy will be at most asymptotically the same (as the total energy bound $E \to \infty$ in such a way that $E/N \to \infty$) as the entropy would have been if the particles were non-interacting.*

Remark. If the 2-body potentials are in addition assumed to have bounded pth moment for any $p > d/2$, then the "3D" box can instead be taken to be any d-dimensional compact region (we assume both here and in the theorem that it has sufficiently smooth boundary, as in footnote 12).

3.7 Relativity

If we similarly consider perturbing a *relativistic* free body Hamiltonian operator R (whose eigenfunctions in a box are again Fourier modes) by adding $-V(\vec{x})$, we can apply the same reasoning to conclude that theorem 6 will still hold with Schrödinger's Hamiltonian $\nabla^2 - V$ replaced by $R - V$. Really this is merely a caricature of relativistic quantum mechanics – it is certainly not realistic to allow potentials acting instantaneously at a distance in a relativistic theory – but we do it anyway. (Proofs of the "stability of relativistic matter" discussed in [32] apply instead.)

3.8 Coulomb case

In particular, consider N Coulombic particles in a 3D box ($n = 3N$), and assume charge neutrality to within $\pm$ a constant number of elementary charges so that $\overline{V}$ (the mean value of the potential, averaged over all possible particle positions in the box^N) grows proportionally to N at most. Then proofs of the "stability of matter" [32] suffice to bound the ground state energy from below. The mean square perturbation of an eigenvalue is bounded by $\overline{V}^2 + CN^2$ where C is the variance of a single Coulomb potential in the box (which, note, is *finite* in 3D in a compact box) in view of additivity of variances (with the charge locations regarded as i.i.d. uniformly distributed random variables), i.e. grows proportionally to N^2. But the square root of this is asymptotically negligible compared with the typical value (proportional to $N^{4/3}$) of the energy of a blackbody radiation gas of N photons in a box [31]. We conclude that allowing Coulomb interacting particles (with approximate total charge neutrality) doesn't help one to build asymptotically larger computer memories for less asymptotic energy cost than just using photons.

[15] One may derive (EQ 23) by integrating first in the planes of the wavefronts, in which we have a circularly symmetric integrand, hence the 2D integral is really 1D, and then perpendicularly to those planes. I admit that the validity of taking the limit as $\theta \to 0+$ to get the Fourier transform of the unscreened Coulomb potential is not clear.

3.9 Alternate proof

Some have expressed skepticism about the second part of my proof of theorem 5, although they have not been able to point out a specific flaw. I feel rather uncomfortable about it myself, because although I can go through the proof of theorem 1 in [12] step by step modifying it as described in our proof, while (as far as I can tell) preserving its validity, the fact remains that I am uncomfortable about my understanding of [12].

The good news is: even if that proof is not valid, we *still* can get our main result by an alternate and in fact simpler (although less powerful) approach. The key idea is to use the "relative infinitesimal boundedness" of the potential energy:

Lemma 7 (Kato: From lemma 4 of [25] and proposition 1 of [27])[16] *Let P and K be the potential and kinetic energy operators respectively for some Coulombic Hamiltonian, or more generally, any Hamiltonian based on interparticle potentials in $L^2(\mathbf{R}^3) + L^\infty$, of N particles. Then the domain D_P of P is a subset of the domain D_K of K and for any constant $\epsilon > 0$, no matter how small, there exists a constant $\zeta > 0$ so that*

$$\|K\Psi\| \leq \epsilon\|P\psi\| + \zeta\|\Psi\| \tag{25}$$

for all $\Psi \in D_P$.

Now Kato [25][17] had shown the validity of the usual Rayleigh-Ritz variational description[18] of the eigenvalues of Schrödinger operators based on $L^2 + L^\infty$ interparticle potentials, i.e. that the mth eigenfunction minimizes[19]

$$\int_\Omega |\vec{\nabla}\Psi|^2 + U(\vec{x})|\Psi|^2 \mathrm{d}\vec{x} \tag{26}$$

subject to the constraints that it be orthogonal to all the previous eigenfunctions and the normalization condition $\int_\Omega |\Psi|^2 \mathrm{d}\vec{x} = 1$. This combined with lemma 7 immediately implies[20] that eigenvalues cannot move by more than a constant additive and an arbitrarily small constant relative amount when K is replaced by $K + P$.

Consequently turning on particle interaction cannot affect the asymptotic behavior of the spectrum, again proving our "ideal gas conjecture."

[16] It is also easy to prove the same relative infinitesimal boundedness statement for the perturbation to the Laplacian operator caused by also turning on bounded time independent smoothly varying *magnetic* fields. (See [22] for a statement of the new operator.) This allows extension of our main result to allow such fields. Although this is a far cry from the ultimate generalization in which the system itself would be allowed to generate possibly enormous magnetic fields, it is a small step in that direction. Recently it was announced [37] that Lieb's proof of the "stability of matter" has been extended to incorporate a magnetism model.

[17] For other dimensions than 3D, see [5]. Here potentials $U(\vec{x})$ in $L^p(\mathbf{R}^n) + L^\infty$ are permitted, where $p = 2$ if $n \leq 3$, and with any $p > n/2$ if $n \geq 4$.

[18] See example 1 section 6.11 of [40]; also [44] gives a more powerful result. In the world of finite dimensional matrices, this is usually called the "Courant-Fischer minimax characterization."

[19] Indeed [25] it will suffice to minimize over linear combinations of the eigenfunctions of the Laplacian in Ω.

[20] There is a simple proof by contradiction.

3.10 Finale

Asymptotically as $E \to \infty$, the maximum number of bits we can store in a box using total energy E and a fixed palette of particles, is at most asymptotic to a constant (arising from the number of particle types) times the entropy (expressed in bits) of a blackbody radiation gas with total energy E filling that box. The known [31] expression for this entropy (but altered to allow $q > 2$) leads to (EQ 1).

An open problem is to improve the present result from an asymptotic one to a more precise bounding statement.

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