A new theorem in vector calculus

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Abstract — **Two well known theorems in 3D vector calculus are Gauss's divergence theorem (actually valid in n dimensions), and Stokes' theorem. We present several formulations of a third natural Theorem of this ilk, which seems to have escaped previous notice.**

Keywords — Stokes, Gauss, divergence theorem, Lorentz force, current loop.

1 Recapitulation of the theorems of Gauss and Stokes

In the below, let all functions, curves, and surfaces be sufficiently smooth[1], and assume all integrals are finite and exist. Assume a right-handed x, y, z coordinate system. For standard vector notation (e.g. $\vec{c} = \vec{a} \times \vec{b}$, $\ell = \vec{a} \cdot \vec{a} = |\vec{a}|^2$) meanings see [1][2]; $\vec{\nabla}$ denotes (in 3D) the differential operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{1}$$

Gauss's divergence theorem states the equality of these two scalars:

$$\underbrace{\iint_V \cdots \int}_{n \text{ integrals}} \vec{\nabla} \cdot \vec{F} \, \mathrm{d}^n \vec{x} \;=\; \underbrace{\iint_{\partial V} \cdots \int}_{n-1 \text{ integrals}} \vec{F} \cdot \mathrm{d}\tilde{\mathrm{A}} \tag{2}$$

where V is some n-dimensional domain, ∂V is its $(n-1)$-dimensional bounding surface, $\mathrm{d}\vec{A}$ is the (outward pointing) vectorial element of surface $(n-1)$-area, and $\mathrm{d}^n\vec{x} = \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \ldots \mathrm{d}x_n$ is the vectorial element of n-volume.
Stokes's theorem states the equality of these two scalars:

$$\iint_D (\vec{\nabla} \times \vec{F}) \cdot \mathrm{d}\vec{A} \;=\; \int_{\partial D} \vec{F} \cdot \mathrm{d}\vec{\ell} \tag{3}$$

where D is a topological disk in 3-space (i.e., a region homeomorphic to

$$\{(x, y) \text{ such that } x^2 + y^2 \le 1\},$$

and ∂D is its bounding curve (homeomorphic to $\{(x, y) \text{ such that } x^2 + y^2 = 1\}$). The infinitesimal element of arc length pointing in the tangent direction to the curve (going clockwise as viewed looking in the $\mathrm{d}\vec{A}$ directions) is $\mathrm{d}\vec{\ell}$.

2 Our new Theorem and its proof

The present paper's **New Theorem** states the equality of these two *3-vectors*:

$$\vec{B} - \vec{C} = \int_{\partial D} \vec{F} \times \mathrm{d}\vec{\ell} \tag{4}$$

where

$$\vec{B} = \iint_D \mathrm{TDD}(\vec{F}) \, \mathrm{d}\vec{A} \tag{5}$$

and

$$\vec{C} = \iint_D \overrightarrow{\mathrm{TDG}}(\vec{F}) \cdot \mathrm{d}A. \tag{6}$$

Here $\mathrm{TDD}(\vec{F})$ is the *t*wo-*d*imensional *d*ivergence of the projected version of $\vec{F}$ (projected down into the 2D tangent plane to the surface D) at the present point in 3-space. Finally $\overrightarrow{\mathrm{TDG}}(\vec{F})$ is the *t*wo-*d*imensional *g*radient (as a 3-vector) of the normal component of $\vec{F}$ (normal to the tangent plane to the surface D, and with the gradient taken in that tangent plane at the current point [of tangency]). Just as in Stokes' theroem, D is a topological disk in 3-space and ∂D is its bounding curve (going clockwise as seen looking along the directions of the normals $\vec{a}$ to the surface D). [See also EQs 7 and 8 below and the reformulation EQ 16.]
Proof sketch: Let $\vec{x} = (x, y, z)$. It suffices to prove it for *linear* functions $\vec{F}(\vec{x})$ only. Also it suffices to prove it merely in the case when D is a *triangle* in 3-space. [The rest will then follow by using the fact that all smooth functions are locally linear; subdividing our arbitrary smooth topological disk into tiny triangles, proof it is OK (in limit of tinyness) to neglect quadratic terms arising from non-flatness of the triangles, non-straightness of the triangle edges, and non-linearity of $\vec{F}$; and cancellation of the 1D integrals on interior triangle edges going both ways to get a $-$ sign cancelling a $+$ sign due to the bilinearity of the vector cross product $\times$ operation.]

Further, due to linearity of all three integrals with respect to $\vec{F}$, it suffices if we prove it only for a suitable

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[1] It will suffice if all surfaces have piecewise continuous unit outward normal vector, all curves have piecewise continuous unit tangent vector, and all functions have continuous derivatives.

set of *basis* functions $\vec{F}$. There are 12 obvious basis functions for the arbitrary linear functions mapping 3-vectors to 3-vectors, namely $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(x,0,0)$, $(y,0,0)$,..., $(0,0,z)$. Finally, by rotational invariance, it suffices if our triangle lies *in a plane parallel to the xy plane.* We now proceed to the details.

(1) If $\vec{F}$ is any *constant* vector the statement is obviously just $0=0$ due to the fact the vector sides of a triangle sum to $\vec{0}$ since the triangle is a closed curve.

From now on by considering adding a constant offset in the z direction we may assume without loss of generality that our triangle lies in the xy plane itself, not just some parallel translate of it.

(2) If $\vec{F}=(0,0,x)$ then the statement comes down to $-\int_D(1,0,0)dA = \int_{\partial D}(-x\mathrm{d}y, x\mathrm{d}x, 0) = (-A,0,0)$ where A is the area of our triangle D (which lies in the xy plane). Similarly if $\vec{F}=(0,0,y)$ and the triangle lies in the xy plane then the statement comes down to $-\int(0,1,0)\times \mathrm{d}\vec{A} = \int_{\partial D}(-y\mathrm{d}y, y\mathrm{d}x, 0) = (0,-A,0)$.

(3) If $\vec{F}=(0,0,z)$ then the statement comes down to $0=0$ (the integrals of $z\mathrm{d}x$ and of $z\mathrm{d}y$ round a closed curve in an xy-parallel plane are both 0).

(4) If $\vec{F}=(z,0,0)$ and the triangle lies in the xy plane then the statement comes down to $0=0$. The right hand 0 is since $\int_{\partial D}(0,-z\mathrm{d}z, z\mathrm{d}y)=(0,0,0)$ if ∂D is a closed curve bounding a topological disk in the xy plane.

(5) If $\vec{F}=(y,0,0)$ then the statement comes down to $(0,0,0)=\int_{\partial D}(0,-y\mathrm{d}z, y\mathrm{d}y)=(0,0,0)$ since $\mathrm{d}z=0$ and integrating $y\mathrm{d}y$ leads to $\pm$ cancellation.

(6) If $\vec{F}=(x,0,0)$ then the statement comes down to $(0,0,A)=\int_{\partial D}(0,-x\mathrm{d}z, x\mathrm{d}y)$ and since $\mathrm{d}z=0$ the first two coordinates are both 0. The integral of $x\mathrm{d}y$ is the area by a slabs argument (slabs $\mathrm{d}y$ wide and x_2-x_1 in width). These 6 cases are the only ones that arise (any others are equivalent) so, **Q.E.D.**

Clearer formulation: How can the new Theorem be formulated purely as math (instead of using English words in the description of the integrals $\vec{B}$ and $\vec{C}$)? Here's one way: Let the surface of a topological disk D be parameterized $(x,y,z)=\vec{W}(p,q)$. Let $\vec{t}=\partial\vec{W}/\partial p$ and $\vec{u}=\partial\vec{W}/\partial q$. Let $\vec{a}=\pm\vec{t}\times\vec{u}$ be a normal 3-vector to the surface D in 3-space so that $d\vec{A}=\vec{a}\mathrm{d}p\mathrm{d}q$. is the infinitesimal element of surface area. (The sign is chosen to make $\vec{a}$ have the correct orientation.) Then the first integral in the Theorem is

$$\vec{B}=\iint_D\left[\vec{t}\cdot\frac{\partial\vec{F}}{\partial p}|\vec{t}|^{-2}+\vec{u}\cdot\frac{\partial\vec{F}}{\partial q}|\vec{u}|^{-2}\right]\mathrm{d}\vec{A}. \tag{7}$$

The second integral is

$$\vec{C}=\iint_D\left(\vec{a}\cdot\frac{\partial\vec{F}}{\partial p}|\vec{t}|^{-2}\vec{t}+\vec{a}\cdot\frac{\partial\vec{F}}{\partial q}|\vec{u}|^{-2}\vec{u}\right)\mathrm{d}p\mathrm{d}q. \tag{8}$$

Remark: Stokes' and Gauss's theorems may be proven in much the same "subdivide into triangles and consider a basis set of linear functions" manner as our new Theorem (only proving them is easier).

3 Confirmatory Examples

Example #1. Let the surface D be the disk $x^2+y^2<1$, $z=0$ and its bounding curve ∂D be the unit circle $x^2+y^2=1$, $z=0$. Let $\vec{F}=(x^2yz^2,\ xyz+3z+9,\ 5x+y^2z+7)$. Then we have

$$\begin{aligned}\vec{F}\times(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z) = ([(xy+3)z+9]\mathrm{d}z-(5x+7+y^2z)\mathrm{d}y,\\ (5x+7+y^2z)\mathrm{d}x-x^2yz^2\mathrm{d}z,\ x^2yz^2\mathrm{d}y-[(xy+3)z+9]\mathrm{d}x)\end{aligned} \tag{9}$$

The integral of this around the unit circle (which is $\int_{\partial D}\vec{F}\times\mathrm{d}\vec{\ell}$) is

$$(0-5\pi, 0-0, 0-0) \tag{10}$$

since everything cancels out (by symmetry or $\mathrm{d}z=0$) except for $\int_{\partial D}-5x\mathrm{d}y=-5\mathrm{area}(D)=-5\pi$. Meanwhile the first surface integral (using as parameters p, q just $p=x$ and $q=y$) is

$$\vec{B}=\iint_D\left[2xyz^2+xz\right](0,0,1)\,\mathrm{d}x\mathrm{d}y \tag{11}$$

which is $(0,0,0)$ by odd symmetry. The second surface integral is

$$\vec{C}=\iint_D(5,2yz,0)\,\mathrm{d}x\mathrm{d}y=(5\pi,0,0). \tag{12}$$

Result: $-(5\pi,0,0)=(0-5\pi,0-0,0-0)$. The Theorem worked.

Example #2. Let $\vec{F}=(x,y,z)$, let the curve be the unit circle $x^2+y^2=1$, $z=0$, and let the surface D be the hemisphere $x^2+y^2+z^2=1$, $z>0$. Then the curve integral is $2\pi\vec{1}_z$. The surface integrals are $\vec{B}=\iint 2\mathrm{d}\vec{A}=2\cdot 2\pi\cdot\frac{1}{2}\vec{1}_z=2\pi\vec{1}_z$ and $\vec{C}=\iint\vec{0}\mathrm{d}A=\vec{0}$ respectively. (In computing $\vec{B}$ we have used the fact that the TDD of $\vec{F}$ is 2, as opposed to $\vec{\nabla}\cdot\vec{F}=3$, we have used the fact the surface area of the hemisphere is 2π, and we have used the fact (due to Archimedes' correspondence between the surface area of a sphere and the cylinder enclosing it) that the average height of the surface of a hemisphere is half its radius. $\vec{C}=\vec{0}$ is since the integrand is everywhere 0 since $\vec{F}(\vec{x})$ is normal to the sphere surface and of constant length on it.) The Theorem worked: $\vec{0}=\vec{0}$.

Example #3. Let $\vec{F}=(0,0,1)$, let the curve be the unit circle $x^2+y^2=1$, $z=0$, and let the surface be the hemisphere $x^2+y^2+z^2=1$, $z>0$. Then the curve integral is $\vec{0}$ by symmetry. The surface integrals are $\vec{B}=\iint 0\mathrm{d}\vec{A}=\vec{0}$ (since the TDD of a constant vector is 0) and $\vec{C}=\iint\vec{0}\mathrm{d}A=\vec{0}$ (since the gradient of a constant vector is 0) respectively, proving once again that $\vec{0}=\vec{0}$.

Example #3 reveals a subtlety: The 2D divergence of $\vec{F}$'s projection into the tangent plane to our surface (TDD($\vec{F}$)), is generally *not* the same as the 2D divergence of $\vec{F}$'s projection onto the surface itself. Similarly, the 2D gradient of the normal-to-plane component of $\vec{F}$ (within that plane, i.e. TDG($\vec{F}$)) is generally *not* the same as the 2D gradient of $\vec{F}$'s normal component to the surface, on that surface. (The former, plane-based quantities are the ones my Theorem wants; the latter surface-based quantities are not. In example #3 the former both are $\vec{0}$, but the latter are both nonzero.)

Example #4. I have, with the aid of the computer-algebra package `MAPLE`, confirmed the Theorem for a *fully general* nonhomogeneous-quadratic polynomial map $\vec{F}(x, y, z)$ from $\mathbf{R}^3 \to \mathbf{R}^3$ in two cases:

1. where D is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$,
2. where D is the flat disk $x^2 + y^2 \leq 1$, $z = 0$.

In both cases ∂D is the unit circle $x^2 + y^2 = 1$, $z = 0$. Also, I tried adding in some (not fully general) *cubic* terms, and the Theorem still passed the resulting tests. In these cases all the integrals are of trigonometric polynomials (if we employ spherical, or polar, coordinates, respectively) hence expressible in closed form.

The full details are too messy to include here, but the `MAPLE` scripts that confirm this are available electronically on my web page[2] and I'll now sketch how it goes for the hemisphere-quadratic. First we define

$$\vec{F}(\vec{x}) = \tag{13}$$
$$(c^{(1)}_{11}x^2 + c^{(1)}_{12}xy + c^{(1)}_{13}xz + c^{(1)}_{22}y^2 + \cdots + c^{(1)}_3 z + c^{(1)},$$
$$c^{(2)}_{11}x^2 + c^{(2)}_{12}xy + \cdots + c^{(2)},\ c^{(3)}_{11}x^2 + c^{(3)}_{12}xy + \cdots + c^{(3)})$$

The curve integral $\vec{I} = \int_{\partial D} \vec{F} \times \mathrm{d}\vec{x}$ may be done by computing $\vec{F} \times \mathrm{d}\vec{x}$ and then making the substitutions $x = \cos\theta$, $y = \sin\theta$, $z = 0$, $\mathrm{d}x = -\sin\theta \mathrm{d}\theta$, $\mathrm{d}y = \cos\theta \mathrm{d}\theta$, $\mathrm{d}z = 0$ and integrating from $\theta = 0$ to 2π. The parameterized hemisphere is $\vec{x} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. We then compute $\vec{v} = \frac{\partial}{\partial\theta}\vec{x}$, $\vec{w} = \frac{\partial}{\partial\phi}\vec{x}$, $\vec{a} = \vec{w} \times \vec{v}$. We now compute the two surface integrals (using EQs 7 and 8 where we are here using ϕ and θ as the parameters, not p and q):

$$\vec{B} = \int_0^{\pi/2} \int_0^{2\pi} \left[\vec{v} \cdot \frac{\partial \vec{F}}{\partial\phi} |\vec{v}|^{-2} + \vec{w} \cdot \frac{\partial \vec{F}}{\partial\theta} |\vec{w}|^{-2} \right] \vec{a} \mathrm{d}\theta \mathrm{d}\phi \tag{14}$$

$$\vec{C} = \int_0^{\pi/2} \int_0^{2\pi} \left(\vec{a} \cdot \frac{\partial \vec{F}}{\partial\phi} |\vec{v}|^{-2}\vec{v} + \vec{a} \cdot \frac{\partial \vec{F}}{\partial\theta} |\vec{w}|^{-2}\vec{w} \right) \mathrm{d}\theta \mathrm{d}\phi. \tag{15}$$

Finally, we confirm that $\vec{I} = \vec{B} - \vec{C}$.

4 3D-only reformulation of the Theorem

Adding $\vec{Q}$ to both $\vec{B}$ and $\vec{C}$ leaves the difference $\vec{B} - \vec{C}$ unaltered. Choose $\vec{Q}$ to be the directional derivative of the component of $\vec{F}$ normal to the tangent plane to D (in the normal direction to that plane) to get this **Slick Reformulation of the Theorem:**

$$\boxed{\iint_D (\vec{\nabla} \cdot \vec{F}) \mathrm{d}\vec{A} - (\vec{\nabla}\vec{F}) \mathrm{d}\vec{A} \ = \ \int_{\partial D} \vec{F} \times d\vec{\ell}.} \tag{16}$$

Here $\vec{\nabla}\vec{F}$ denotes the 3×3 matrix whose i-down j-across entry is $\frac{\partial}{\partial x_i} F_j$. It multiplies the column-vector $\mathrm{d}\vec{A}$.

EQ 16 has the advantage that it is formulated purely in terms of the usual 3D differential operator $\vec{\nabla}$ rather than our invented 2D-inside-3D operators TDD and $\overrightarrow{\mathrm{TDG}}$. [On the other hand, in some applications, the original 2D-in-3D formulation might be more advantageous. I have successfully tested example #4 (extended with cubic terms) for the reformulated theorem too.]

5 How to view it as Stokes in disguise

At first I suspected the present Theorem was only a tiny consequence of an ultra-general theorem of Poincare [3] about differential forms on manifolds. Poincare's theorem subsumes both Stokes' theorem and the divergence theorem on n-manifolds as special cases. But that cannot be directly true because (on our 2-manifold D) it involves 3-vectors rather than 2-vectors; and any differential form on an n-manifold has some power of n components, but 3 is not a power of 2. That led me to believe that the present result really is new.

But the reformulation in §4 suggested that the new Theorem is really just Stokes' theorem used 3 times with results linearly combined, with various altered functions employed inside the different Stokes invocations. This turns out indeed to be true; in a conversation with Yury Grabovsky (Temple Univ. Math. dept.) we were able to produce such a **second proof.**

Here it is. Consider the jth component of EQ 16, i.e. (letting $\vec{e}_j$ denote the unit vector in the x_j direction)

$$\int_{\partial D} \vec{e}_j \cdot (\vec{F} \times \mathrm{d}\vec{\ell}). \tag{17}$$

Using the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ this is

$$= \int_{\partial D} (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{\ell}. \tag{18}$$

Applying Stokes' theorem this is

$$= \iint_D \vec{\nabla} \times (\vec{e}_j \times \vec{F}) \cdot \mathrm{d}\vec{A}. \tag{19}$$

Now employing the vector identity ([1] 10.31#7)

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} \tag{20}$$

and taking advantage of the facts that $\vec{\nabla} \cdot \vec{e}_j = 0$ and $\vec{\nabla} \times \vec{e}_j = \vec{0}$ since $\vec{e}_j$ is a constant vector, this is

$$= \iint_D \left[\vec{e}_j(\vec{\nabla} \cdot \vec{F}) - (\vec{e}_j \cdot \vec{\nabla})\vec{F} \right] \cdot \mathrm{d}\vec{A} \tag{21}$$

$$= \vec{e}_j \cdot \iint_D (\vec{\nabla} \cdot \vec{F}) \mathrm{d}\vec{A} - (\vec{\nabla}\vec{F}) \mathrm{d}\vec{A} \tag{22}$$

proving the Theorem. **Q.E.D.**[3].

[2] `http://math.temple.edu/~wds/homepage/works.html`

[3] Of course all results in real analysis depend on the same set of Axioms of real numbers, hence are not independent except in the extremely rare cases that they depend on disjoint axiom subsets. So the question of "newness" is a subjective one.

6 Consequences in electromagnetism

The right hand side of our Theorem is of course a very natural vector quantity, which arises in electromagnetism when computing the **Lorentz force** exerted by a magnetic field $\vec{F}(\vec{x})$ on a loop of wire ∂D circulating an electric current.

One immediate **corollary** of our Theorem's 3D-only reformulation in §4, is the following. Suppose a magnetic field $\vec{F}(\vec{x})$ obeys the Maxwell equation

$$\vec{\nabla} \cdot \vec{F} = 0 \quad \text{(no magnetic monopoles)} \tag{23}$$

Suppose D is a topological disk surface in $\mathbf{R}^3$ which is such that, at all points $\vec{x} \in D$, the component of $\vec{F}(\vec{x})$ normal to D is *constant*. Then the Lorentz force on the current loop ∂D is zero: $\int_{\partial D} \vec{F} \times \mathrm{d}\vec{\ell} = \vec{0}$.

References

[1] I.S. Gradshteyn & I.M. Ryzhik: Table of Integrals, Series, and Products, Academic Press. Chapter 10 is on vector identities and vector calculus.

[2] J.D.Jackson: Classical electromagnetism, Addison-Wesley 3rd edition 1998.

[3] M.Spivak: A comprehensive introduction to differential geometry, Publish or Perish, 5 volumes, ≈1979.