Odd squared distances

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Abstract — **We prove the following theorem: In n dimensional euclidean space, the largest set of points that exists so that every squared distance between any point pair is an odd integer, has cardinality $n+2$ if $n \equiv 2 (\mathbf{mod}\ 4)$ and $n+1$ otherwise.**

Keywords — odd integer distances, Laplace identity, Sommerville determinant.

A related previous **theorem:** *In n dimensional euclidean space, the largest set of points that exist so that every distance between any point pair is an odd integer, is exactly: $n+2$ if $n \equiv 14 \pmod{16}$, $n+1$ otherwise.*

This related theorem is due to R. L. Graham, B. L. Rothschild, and E. G. Straus: *Are There $n+2$ Points in E^n with Odd Integral Distances?*, Amer. Math. Monthly, 81 (1974), 21-25. We consider squared distances to be somehow more fundamental than distances, which is why this paper is here. For example, points with integer coordinates have integral squared distances but not (generally) integral distances.

A trivial but related **theorem:** *It is impossible to have 3 points in $\mathbf{Z}^n$ (i.e. with integer coordinates) with only odd squared distances.*

The trivial **proof:** One of the points is $(0, 0, ..., 0)$ WLOG, so the other two points each have an odd number of odd coordinates. Two points having an odd number of odd coordinates cannot have an odd squared distance between them. QED.

Proof of the main theorem (stated in our abstract): Obviously $n+1$ points are always possible, e.g. forming a regular simplex.

Suppose that $n+2$ points were possible. Let $d(i, j)$ be the distance from point i to point j and $d(i, i) = 0$. The Cayler-Menger-Sommerville determinant, i.e. the determinant of the $(n+3) \times (n+3)$ matrix S whose ijth entry is $S_{ij} = d(i, j)^2$ for $i, j \in \{1..n+2\}$ and with the off-diagonal entries in the $(n+3)$rd row or column are 1, and the diagonal entries $S_{ii} = 0$, is zero. (Because this determinant is proportional to the squared volume of the simplex in $n+1$ space. See D.M.Y. Sommerville: *An introduction to the geometry of n dimensions,* Dover 1958.)

Now, evaluate this determinant mod 2. The matrix viewed mod 2 has all off-diagonal entries = 1, and all on-diagonal entries = 0. Since this matrix is circulant, its eigenvectors are complex exponential waves, and the corresponding eigenvalues are each -1, except that the trivial wave $(1, 1, 1, ..., 1)$ has eigenvalue $n+2$.

So the value of the determinant is $(-1)^{n+2}(n+2)$. Mod 2 this is n. This is not zero if n is odd.

This proves the main theorem in the case when n is odd. It also proves, for any n, that the most points that are possible is $n+2$.

Note: If the distances rather than the squared distances are required to be odd integers, then since odd squares are 1 (mod 8), evaluating the determinant above mod 8 gives $(-1)^{n+2}(n+2)$ mod 8, which proves the "related previous theorem" in the cases when $n \not\equiv 6 \pmod 8$.

Now suppose $n \equiv 2 \pmod 4$, so that $n = 2m$ with m odd. Consider these $n+2$ points in n-space:

$$(0, 0, 0, ..., 0, K, 0, ..., 0)/\sqrt{2} \quad (n \text{ points})$$
$$(A, A, A, .., A)/\sqrt{2}$$
$$(B, B, B, .., B)/\sqrt{2}.$$

The squared distances among them are

$$K^2,$$
$$mA^2 + (K-2A)K/2,$$
$$mB^2 + (K-2B)K/2,$$
$$m(A-B)^2.$$

We want all these things to be odd integers. Choose p so that $K = mp$ is an odd integer, choose q so that mq is an odd integer, and let $A = (p+\sqrt{q})/2$, $B = (p-\sqrt{q})/2$, $K = mp$. Then K^2 and $m(A-B)^2$ are automatically odd integers. In the subcase $m = 4x+1$ we choose $q = 3$, $p = 1$ and find for the middle two squared distances $8x^2 + 6x + 1$. In the subcase $m = 4x+3$ we take $q = 7$, $p = 1$ giving $8x^2 + 18x + 9$ for the middle two squared distances. Thus, by explicit construction, in dimensions $n \equiv 2 \pmod 4$, there always exist $n+2$ points with odd squared distances.

This proves the main theorem in the case when $n \equiv 2 \pmod 4$.

Finally, we **claim** the determinant (which has to be 0 for $n+2$ points to exist in n dimensions with all squared distances odd) is always $\pm 2 \pmod 8$ if n is divisible by 4, hence these dimensions cannot have $n+2$ points. Also,

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if $n \equiv 2 \pmod 4$ then the determinant is always 0 (mod 4), i.e. 0 or 4 (mod 8).

These mod 8 claims may be proven as follows by a double induction. We already know the statement is true if all nonzero entries of the matrix (mod 8) are 1. That is a basis for the induction. Assume it is true for all $(n+3) \times (n+3)$ matrices S (mod 8) we have considered so far. Now, consider changing a nonzero entry to 1, 3, 5, or 7 (and its symmetrical entry too). The change in the determinant is always divisible by 4. This may be proven via a double row and column deletion and expansion by 2nd order minors perturbation argument – and the inductive assumption of the claim in $n-2$ dimensions. (There is a famous formula by Laplace which generalizes the "expansion by minors" identity suitably, e.g. see P.Lancaster and M.Tismenetsky: *The theory of matrices with applications,* 2nd ed., Academic Press 1985, section 2.4.) This allows us to inductively enlarge the set of "matrices we have considered so far" one by one until all are covered.

That proves the main theorem in the sole remaining case $n \equiv 0 \pmod 4$. QED.

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