Range voting

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Abstract —
The "range voting" system is as follows. In a c-candidate election, you select a vector of c real numbers, each of absolute value ≤ 1, as your vote. E.g. you could vote $(+1, -1, +.3, -.9, +1)$ in a 5-candidate election. The vote-vectors are summed to get a c-vector $\vec{x}$ and the winner is the i such that x_i is maximum.

Previously the area of voting systems lay under the dark cloud of "impossibility theorems" showing that no voting system can satisfy certain seemingly reasonable sets of axioms.

But I now prove theorems advancing the thesis that range voting is uniquely best among all possible "Compact-set based, One time, Additive, Fair" (COAF) voting systems in the limit of a large number of voters. ("Best" here roughly means that each voter has both incentive and opportunity to provide more information about more candidates in his vote than in any other COAF system; there are quantities uniquely maximized by range voting.)

I then describe a utility-based Monte Carlo comparison of 31 different voting systems. The conclusion of this experimental study is that range voting has smaller Bayesian regret than all other systems tried, both for honest and for strategic voters for any of 6 utility generation methods and several models of voter knowledge. Roughly: range voting entails 3-10 times less regret than plurality voting for honest, and 2.3-3.0 times less for strategic, voters. Strategic plurality voting in turn entails 1.5-2.5 times less regret than simply picking a winner randomly. All previous such studies were much smaller and got inconclusive results, probably because none of them had included range voting.

Keywords — Approval voting, Borda count, plurality, uniqueness, social choice, Condorcet Least Reversal, Gibbard's dishonesty theorem, strategic voting, Monte Carlo study, Bayesian regret, voter ignorance.

Contents

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1 Generic and non-generic elections

Before considering range voting, it will help to illustrate some generalities about voting by looking at the most popular voting system (albeit a poor one if $c \geq 3$): plain plurality[1]. Here each voter awards 1 vote to one of the c candidates and 0 votes to each of the remaining $c-1$. The winning candidate is the one getting the most votes. (Ties are broken randomly.)

In a ($c = 2$)-way election with V voters, each of whom is modeled as flipping a fair coin, the probability that your vote will have an effect (i.e., break a tie, or cause a tie which gets broken your way) is asymptotically (as $V \to \infty$) $1/\sqrt{2\pi V}$ if V is even, and 2 times larger if V is odd. (The odd-V formula arises since $\binom{2n}{n}4^{-n}$, which is the probability of a tie among the $2n$ voters besides yourself, is asymptotic, as $n \to \infty$, to $1/\sqrt{\pi n}$, by "Stirling's formula." Here $V = 2n+1$. The even case is similar.) If we average over the parity of V, this is $3/\sqrt{8\pi V}$. Thus in a $\approx 10^6$-voter election, the probability your vote will have an effect would be $\approx 1/1671$. So if it costs you 1 dollar (e.g. in wasted time and transport costs) to vote, it is not worth it for you to bother to vote, if the outcome of the election will affect you by ≤ 1671 dollars.

But the situation is far worse in what I will call the *generic case*, where the voters are modeled as tossing *biased* coins. Say the coin comes up heads with some fixed probability p. (The above formula was for the special case $p = 1/2$, but generically $p \neq 1/2$.) In that case, as $V \to \infty$, the probability that your vote will have an effect, declines *exponentially*: it is about a factor $(4[1-p]p)^{V/2}$ times smaller than the formula for the $p = 1/2$ special case. For example if V is odd ($V = 2n+1$) then the probability your vote will have an effect is $\binom{2n}{n}p^n(1-p)^n$. Thus in a 10^6-voter election with 51-49 favortism of one of the 2 candidates, the probability your vote will have an effect is only about 10^{-90}. In this case, even if the fate of the entire universe hinged on the outcome of the election, and you only had to sacrifice one atom of money in order to vote, it still would not be worth bothering, since the universe contains $< 10^{80}$ atoms.

Thus it is usually irrational for most people to vote. In the generic case it is irrational for anybody to vote. Therefore the only voters are either irrational, or those few with a tremendous personal stake in an ultra-close election, or those who have been bribed to vote. If it is regarded as better for society that large elections be decided by rational unbribed voters, then *it is essential to make voting as cheap and simple as possible*, for example by allowing voting by telephone or over the internet. Any scheme, therefore, that asks voters to vote more than once per election (such as the present USA system of "primary" and "secondary" elections, or the French "plurality plus runoff" system with a "second round"[2]), or which is in the slightest way complicated, should be ruled out[3].

These (exponential and inverse square root) dependencies of voting power on the number V of voters are quite different from what one might naively have expected, namely $1/V$.

But, now consider an election in which the coin-probability p is itself a random variable, uniform in $[0,1]$. In that case, if the number of voters V is odd, $V = 2n+1$, the probability your vote will have an effect is

$$\int_0^1 \binom{2n}{n} p^n(1-p)^n \mathrm{d}p. \tag{1}$$

This integral is just an "Euler beta function," so we may evaluate it in closed form. The result is $1/V$. Mathematical poetic justice has prevailed!

For an even number of voters, $V = 2n$, this becomes

$$\frac{1}{2}\int_0^1 \binom{2n-1}{n-1} p^{n-1}(1-p)^n \mathrm{d}p \tag{2}$$

assuming you plan to vote "yes;" and that the probability is $1/2$ that any tie you create is broken your way. (If you planned to vote "no" then this expression with p changed to $1-p$ would be used, which would have the same value after integrating.) This integral has value $1/(2V)$, which is not quite so poetic.

Summary: There are two kinds of 2-candidate, V-voter elections: "generic" ones (which happen almost all the time) in which your voting power (probability of affecting the election result) declines exponentially with V, and rare "nearly tied" ones in which your voting power only declines like $3/\sqrt{8\pi V}$. The net effect (averaged over

[1] Plurality is called "first past the post" in Britain.

[2] In which the top two candidates from the first round (if nobody got $> 50\%$ in that round) are voted on

[3] It has been pointed out to me that many people still vote, and they are not all crazy. (Turnouts are presently about 50% in presidential election years and about 35% in nonpresidential election years, in USA elections, and they seem to be declining with time.) This could be viewed as a good counterexample to the foundational notion of economics that people are "rational," i.e., strive to increase their own wealth. Or it could be viewed as a good reason the present paper's later analyses of the effects of "rational voting" should not be taken too seriously – at least, no more seriously than the arguments of economists. On the other hand, there clearly is some evidence for effects caused by rationality in voting behavior. For example the well known dominance of the "two party system" in the USA, is obviously the cumulative result of the fact (cf. §2, "tactical thinking") that a rational voter in large plurality elections generically will always vote for one of the two frontrunners in the polls, regardless of the merits of the remaining candidates. (After 1824, when most presidential candidates were unaffiliated, every presidential election has been won by a member of one of the two major parties. There were, however, several times when a major party broke up and re-formed, or was renamed. For example in 1864 the Republican party's name was temporarily changed to the "Union" party. In ≈ 1834 the "National Republican" party broke up, with most members joining the new "Whig" party; by 1854 the Whigs in turn had dissolved and most of its Northern members had joined the new "Republican" party. Since then every US president has been either a Republican or a Democrat.) This phenomenon has been called "Duverger's law" [13].

both kinds of election and over the parity of V), is that one's voting power is exactly $0.75/V$.

Definition. A nonnegative quantity is "exponentially small" if it is bounded by k^V for some fixed k, $0 < k < 1$, for all sufficiently large numbers V of voters.

Definition. An event is "generic" if it happens in a full-measure subset of some real parameter space, e.g. the event "$p \neq 1/2$" is generic in the space $0 \leq p \leq 1$.

The whole notion of a "generic" election, in which every interesting probability and probability ratio goes exponentially to 1 or 0 as $V \to \infty$, is going to be key throughout this paper. I will take the attitude that

1. Voting systems should be analysed in generic elections.
2. Voting systems with unsatisfying behavior generically, should be eliminated from consideration.
3. Only then should nongeneric behavior be considered.

Admittedly, the analysis I've just given has only been for the plurality voting system, but it should be evident that these same exponential phenomena occur, assuming independent voter vote probabilities, in many other voting systems too, for example all the COAF systems (§3) and in systems with a finite number of rounds, whose individual rounds are COAF. It seems pointless to try to elucidate exactly for which voting systems these exponential trends happen, since it always seems to be trivial to decide any particular case.

2 Tactical thinking that follows from these exponential properties

In plurality elections having $c \geq 3$ candidates, "tactical" considerations prevail[4]. Here the thinking is that one "must" vote for one of the top two candidates, otherwise one's vote is "wasted." We can now see that this thinking is entirely justified, because, generically, a vote for anybody other than the two frontrunners has an exponentially smaller chance of having an effect. In short, even if you think non-frontrunner N is a million times better candidate than frontrunner G, who in turn is only 1.001 times better than frontrunner B, then, because N's chances of being elected are at least 10^{90} times smaller than either G's or B's chances, your best move is to "dishonestly" vote for G.

Generically, the chances for each candidate successively lower ranked in the pre-election polls, are smaller by exponentially enormous factors than those of the previous candidate.

Incidentally, due to the importance of such tactical thinking, later voters (e.g. Hawaiians) have more power (and more incentive to vote dishonestly) if tallies of earlier votes (e.g. by voters in Maine, which has earlier time zones) are made public. If the goal of the election is "democracy" (in which all voters have equal power, and are encouraged to express their honest opinions when they vote) then no reporting of vote totals should be permitted until the election is over.

The purpose of this paper is to claim that there is a better voting system available: "range voting." In range voting there is no "tactical disadvantage" to voting for your favorite candidate. We will furthermore see that range voting is uniquely optimum among a wide class of possible voting systems, in the sense that it allows voters to provide the "most possible information" without being "tactically misguided."

3 Compact set based, One-vote, Additive, Fair systems (COAF)

Most of this paper will restrict itself to the following (wide) class of voting systems:

Let there be c candidates and V voters, c fixed, V large. Each voter chooses, from a fixed **C**ompact set $S \subset \mathbf{R}^c$ of "allowed votes," **O**ne c-vector. The vectors are **A**dded. The maximum entry in the summed c-vector corresponds to the winner.

Such a system is "**F**air" if S is invariant under the group of $c!$ permutations of the c coordinates of $\mathbf{R}^c$.

Here is a list of several COAF voting systems, along with the descriptions of the corresponding sets S.

1. Plurality: S is c vectors, each of shape $(+1, 0, 0, \ldots, 0)$.
2. Bullet: S is c vectors, each of shape $(-1, 0, 0, \ldots, 0)$.
3. Borda[5] [31][32][36]: S is the $c!$ permutations of $(c-1, c-2, \ldots, 2, 1, 0)$.
4. Dabagh [11]: S is $(c-1)c$ vectors, each of shape $(1, 1/2, 0, 0, \ldots, 0)$.

[4] Example: in an editorial on 26 October 2000, the *New York Times* advised "tactically minded" voters to vote for Gore even if their favorite candidate was Nader. It even went so far as to call Nader's candidacy a "disservice" because it might siphon off enough Gore voters to result in Gore's defeat. Two weeks later, in the closest US presidential election ever, Gore indeed appeared to have lost the election – due to the sub-election in Florida in which indeed, the difference between Gore's and Bush's vote totals was about 9×10^{-5} of the total votes, and also $< 1\%$ of the 97488 Nader votes in Florida. (Polls indicated 40% of Nader voters would have voted Gore, and 20% Bush, had Nader not run.) Despite this, in neither this nor in any other US presidential (or Senatorial, or Gubernatorial) election, has anybody's single vote ever had any effect, since no US presidential election (or statewide sub-election) has ever been tied – at least according to claims in the *New York Times* on Sunday, 12 November 2000. In that article it was claimed that a vote recount showed the statewide New Mexico election was won by G.W.Bush over A.Gore by 4 votes (Gore had won the original vote), which the *Times* called the closest presidential state vote ever (claiming the 2nd closest was T.Roosevelt winning Maryland in 1904 by 51 votes). But later it was claimed Bush had won by 17 votes. Then [Albuquerque Journal, 15 Nov] it was claimed this small margin for Bush was actually due to a handwriting error during the counting, so that *Gore* had a 375-vote margin. By 18 Nov, 375 had transmuted to 481, and the transmutations continued, ultimately reaching 366.

[5] Some people also allow "truncated preference" vote-vectors, such as $(3, 2, \frac{1}{2}, \frac{1}{2})$, in (their version of) Borda.

5. Approval [6]:
   S is 2^c vectors of shape $(\pm 1, \pm 1, \ldots, \pm 1)$. (But 2 of these 2^c possible vote-vectors, namely those with all coordinates equal, are "silly.")

6. Range: S is the cube $|x_1| \le 1, |x_2| \le 1, \ldots, |x_c| \le 1$.

Note in the "bullet" system you vote *against* a most-hated candidate. In the Borda system you rank-order the candidates and the entries of all the permutation-vectors are summed. Also Borda is equivalent to the aggregate sum of all $\binom{c}{2}$ pairwise elections (assuming voters are honest in, and vote in, each pairwise election). In the Dabagh "vote and a half" system you give a first-choice candidate 1 vote and a second-choice candidate half a vote. If the number of candidates $c = 3$, then Dabagh is equivalent to Borda. If $c = 3$, then S_{Borda} also is equivalent to the 6 vectors of shape $(+1, 0, -1)$, hence is essentially equivalent to the sum of a Bullet and a Plurality vote.

3.1 Non-COAF voting systems

Unfortunately, not every known voting scheme is COAF. Counterexamples:

1. Thomas Hare's (1850) "Single Transferable Vote" (STV) system [1] in which each voter specifies a c-permutation as his vote. The system eliminates the candidate with the fewest first-place votes from consideration, thus reducing all c-permutations to $(c-1)$-permutations, and this is repeated until, after the $(c-1)$th round, only 1 candidate (the winner) remains[6].

2. Schemes based on "runoff" sub-elections either are not "one-vote" or are not additive (i.e. depend on something other than who has the largest summed vote) or are not fair, so COAF would also disallow them.

3. The "electoral vote" scheme used in the US is not "additive" and so COAF would disallow it. (It is also unfair in the sense that it is not invariant under a permutation of the *voters* – as opposed to our usual definition of "fairness" involving permuting the candidates.)

4. "Least reversal voting" (similar to a system advocated by the Marquis de Condorcet in 1785): In this system, each vote is a rank-ordering of the c candidates (perhaps with ties, e.g., $A > B > C = D > E$). Consider the arc-weighted directed graph, with c nodes, where the arc $i \to j$ is directed toward the winner of the pairwise i versus j election, and is weighted by the margin (difference in number of votes) of that defeat. If this digraph has a unique node with outvalence zero, then that is the winner. Otherwise, a set of arcs of minimum summed weight, is chosen, such that by reversing their directions, one gets such a winner. This scheme also can be modified to choose w out of c "winners;" we reverse the set of arcs of minimum summed weight so that a subset of exactly w nodes is created, from which the remainder of the digraph is unreachable. Also it can be modified to produce a full ordering of all c candidates: we reverse the set of arcs of minimum summed weight so that the digraph becomes acyclic [37][7].

5. The "Copeland system" which is like Condorcet, except the winner is simply the winner of the most pairwise elections, ignoring the margins in each.

6. Black's suggestion [3]: Start to use Condorcet, and if a candidate exists who won *every* pairwise election, then he wins. Otherwise, abandon Condorcet and use Borda.

7. The "Bucklin system:" The voters supply a rank ordering of the candidates as their vote. If a candidate has a "Majority of the Voters" (i.e. a number greater than one half of the total number of voters) from the count of the first choices, then that candidate wins. If no candidate has a majority, the second choices are added to the first choices. If one or more candidates now has a "Majority of the Voters," the candidate with the greatest number wins. (A "Majority of the Voters" is not the same as a majority of the sum now being used.) If there is no such candidate, then the third choices are added to

[6] Essentially this system is used in Irish elections and to elect Australian representatives. In Britain it (when used to elect a single winner) it is called the "Alternative Vote." It has also been called "instant runoff voting" (IRV). (The thinking behind that: "Runoff" elections in which a revote is made for the top candidates only from a previous election with more candidates, supposedly are superior to plurality elections when there are more than 3 candidates. STV systems give the effect of having a runoff without forcing voters to vote more than once, hence supposedly are even more superior.) Hare's STV system can also be used to elect any number w, $1 \le w < c$, of "winners." One disadvantage of this system is non-monotonicity: ranking a candidate higher in one's vote can actually decrease his chances of winning. There is also Clyde H. Coombs's variant STV system in which the candidate with the largest number of last-place votes is eliminated each round. It seems obvious that Coombs's system is far worse than Hare's because it is far more easily destroyed by manipulation: Specifically, all rational voters will vote their least-liked frontrunner candidate an artificial "last place" ranking. (Meanwhile, honest opinions about who, among the large number of candidates, is the worst, would be split among the many deserving choices.) In this way, all frontrunners would be essentially certain to be eliminated in early rounds and some unknown "dark horse" would always be elected. However, see §10.7 theorem 8 for some distressing properties of *Hare's* system under certain voter strategies. In yet another possible STV system, the candidates, not the voters, select and announce to whom their vote is to be transferred if they are eliminated – but that idea is not possible if the "candidates" are not people, just alternatives.

[7] The motivation behind these: if we assume there is a true best candidate ordering, or a true best candidate, or a true best w-subset of candidates, and each voter is modeled as outputting each preference relation $A > B$ with a fixed probability $p > 1/2$ that it corresponds to the truth (and all probabilities are independent) then the likelihood of the given votes is maximized by the order, or w-subset, the Condorcet-LR procedure determines.

the combined first and second sums. This continues until a "majority" winner can be declared[8].

Additive schemes have simplicity advantages, such as the fact that it is not necessary to remember the individual votes (one merely needs to keep track of a c-vector "running total"). With an additive scheme, each voting location can transmit its subtotal to the central agency, whereas in nonadditive systems (e.g. transferable vote), it may be necessary to transmit a far larger amount of information, which might make problems and mistakes more likely. (See **Wk** discussion in §9.3.)

4 Honest Voters and Utility Voting

Definition. Some event E is "best for society" if it maximizes (over the possible alternatives E', E'',... to E) the sum, over all voters (members of society) v, of the utility for v of E.

If all voters were "honest" then the clearly best voting scheme would be: each voter votes $(U_1, U_2, ..., U_c)$ where U_i is the utility for that voter of "candidate i gets elected." (This *would* have been COAF with $S = \mathbf{R}^c$, *except* that this S is not compact.) Then the winner would automatically maximize overall summed Bayesian utility for all society.

But... obviously, our knowledge of human nature tells us that assuming such honesty is ridiculous; all it would take to destroy everything would be for just one dishonest voter to claim an enormous utility for some candidate[9]. (For *robot* voters, such assumptions of honesty need not be ridiculous[10].)

A more robust variant of the same system would be "scaled utility voting." Here, each voter awards each candidate a number in $[-1, 1]$ of votes, where the maximum and minimum utility (if elected) candidates for that voter would get $+1$ and -1, and the others would be linearly interpolated between ± 1 according to their utility values. Although it still is unrealistic to assume voter honesty in such a system, at least the damage a small number of dishonest voters can wreak, is bounded. Furthermore, if it were the case that $\max_i U_i - \min_j U_j = 2$ for each voter (so that no scaling was required), then honest scaled utility voters would indeed elect the genuinely best candidate for all of society.

5 Rational Voters and what they'll generically do in COAF voting systems

Consider "maximally rational but dishonest" voters. (For short I will just call them "rational.") A rational voter picks the vote from S which maximizes his expected utility given that he has seen pre-election poll results for a small random subset of the other voters and given his utility values $\vec{U} = (U_1, U_2, \ldots, U_c)$ for all the c candidates. We shall assume the voter shall assume that these poll results are a fairly accurate prediction of what is likely to happen in the upcoming election, i.e., that the pollees did not lie to the pollsters[11].

Definition. Our model of an election will be this 4-step process:

1. A non-binding pre-election poll is performed, consisting of an election (under the same voting system that will later be used in the real election) among a small random subset of the voters.

2. The results of that poll are publicized, in the form of a Gaussian probability distribution predicting the election result (assuming votes won't change) whose mean and covariance matrix are derived from the poll-sample vote vectors.

3. Election is conducted, yielding a single winner w.

[8] Bucklin was used in 7 US states starting in 1912 in votes for important offices, including gubernatorial races. It is mentioned in [11]. Bucklin was found to be (and is) defective in that a voter's second-choice vote often can help to defeat that voter's first-choice. The system is thus non-monotonic. Hence many voters are motivated to lie about their second choices, or motivated simply not to indicate a second choice (and, empirically, they mostly indeed did single-vote). But this thwarted the goal of discovering which candidate was favored by a majority of voters. Hence all US states eventually dropped Bucklin. A 2-round Bucklin system (essentially) is presently in use in the London Mayor's election (where it is called the "supplementary vote").

[9] Thus, this voting system should only be adopted when it is known that the voters are people of the utmost honesty and dependability. For example, in the world at present, it is only used in one place I know of: judging the winning fighter in boxing matches.

[10] For example, suppose there are 77 independent algorithms that attempt to recognize a video image as corresponding to one of 100 human faces. How can we decide, based on the 77 algorithm outputs, which of the 100 humans is the most likely to be the correct answer? A reasonable solution would be to employ honest utility voting, with each algorithm's "utilities" being its estimated log-likelihoods for each of the 100 humans. "Honesty" could be assured by generating the 7700 log-likelihood estimates using a common procedure based on recorded historical performance for each of the 77 algorithms.

[11] However, rational pollees *would* lie to the pollsters. Furthermore, if there were a sequence of such polls, say, 1 per week before the election, then rational voters would presumably try to lie to the pollsters, probably in different ways in each poll, in an attempt to manipulate people's "strategic voting decisions" during the upcoming election – as well as in an attempt to manipulate people's "strategic lying decisions" in future polls! I am going to ignore all that and assume that rational voters will assume that the poll results are a good approximation to what the vote vectors's mean (and covariance matrix) will be. Experiments [18] in artificial elections in which volunteers were paid amounts that varied depending on the election result, indicate (agreeing with my assumptions) that voters *do* change their votes, strategically, in response to information coming from pollsters. These experiments *also* suggest (*dis*agreeing with the present paper's assumptions) that voters will learn to *lie* strategically to the pollsters. However, I do not think that invalidates the present paper because these experiments were highly unrealistic – every voter was polled before every election, every voter knew when they were going to be polled, and the entire polling&voting cycle was repeated many times each day by each volunteer. Realistically, typical voters go their entire lives without ever being polled and hence do not bother to precalculate what the best lie to tell the pollster is because the probability they will be polled is very small (and they do not have enough time to calculate that lie immediately in the absence of precalculation). This is presumably why it seems experimentally justified, in countless real elections, that polls are accurate predictors.

4. Each voter v now receives a utility-payoff $U_w^{(v)}$ depending on the identity w of the winner.

The poll results essentially necessarily *must* be of the following form (assuming all pollees make their decisions about what vote vector to supply to the poller independently – which is reasonable in the absence of communication between them): they will tell us the mean c-vector and variance $c \times c$ matrix of a c-dimensional *normal* distribution[12]. Consider the limit $V \to \infty$. Affinely transform $\mathbf{R}^c$ to make the Gaussian be spherical with center at $\vec{0}$. The decision hyperplanes $x_i = x_j$ are still hyperplanes in the affined space A. The probability the election will cause a candidate k to win, is the integral of the Gaussian over the part of A where k won. The probability your vote can have an effect, is, essentially, the integral of the Gaussian over a thin slablike neighborhood of all the decision surfaces. Because the Gaussian is sharply peaked (having $\sqrt{\text{variance}}$ of order $\sqrt{V}$, which is far smaller than the |mean|, which is of order V), this integral is, generically, exponentially unaffected by anything other than the *closest* decision hyperplane surface to $\vec{0}$, when $V \to \infty$.

So the **procedure the rational voter will adopt** is:

1. In the affined space A, find the decision surface closest to $\vec{0}$.
2. Let i and j be such that that surface corresponds to candidate i versus j.
3. *Generically,* the voter will then vote the vector in S furthest in the direction $\pm(\vec{e}_i - \vec{e}_j)$ (where $\vec{e}_k$ denotes the unit vector in the x_k direction) This is because the probability the election will be won by i or j is, generically, enormously exponentially close to 1 for V large. The voter is best off trying to alter who of i or j wins. If he devotes any of his vote to altering a versus b where $\{a, b\} \neq \{i, j\}$ then the probability that that part of his vote will have any effect[13] would be enormously exponentially smaller and hence could not (in the limit $V \to \infty$ with his utilities $\vec{U}$ held fixed) maximize expected utility.

Definition. A vote $\vec{v}$ "provides information about candidate k" if there exist two votes $\vec{v}$ and $\vec{v}'$ that voter could have supplied, with $v_k \neq v'_k$.

Given that this is what is going to happen... *there is only one circumstance in which the* rational *voter would ever output a vote giving any information about his preferences on any candidates besides i and j*: if the set S happened to have a "flat face" normal to the direction $\vec{e}_i - \vec{e}_j$! (Note: if S were a sphere, or anything with a curved surface locally behaving polynomially, then still the rational voter would only output an exponentially small amount in the non-i, non-j coordinates of his vote. So S really needs to have a super-polynomially *flat* face to make the rational voter output non-negligible information, and hence in the limit $V \to \infty$ this face must become genuinely flat.)

One can similarly argue that, once the rational voter decided he wanted to be on one given face (say the hyperplane $x_i - x_j = \text{const}$) of S, then, within that face-plane, we could perform a similar affined Gaussian argument in 1 dimension less, and thus he would again act to move as far as possible in some direction $\vec{e}_a - \vec{e}_b$ for some a, b with $\{a, b\} \neq \{i, j\}$. Hence, to force (or allow) the rational voter to continue to provide information after he's provided information about candidates i, j, a, b, the boundary of this face must *also* be flat and oriented normally to $\vec{e}_a - \vec{e}_b$. (See §7 for more discussion of rational voter strategy in COAF systems.) We can continue on in this way to argue that every relevent boundary of every face of every dimension of S, must be determined solely by hyperplanes of the form $x_i - x_j = \text{const}$.

6 A particular natural compact set P

It is now natural to consider the Polyhedral set P in $\mathbf{R}^c$ defined by

$$P = \{\vec{x} \in \mathbf{R}^c \text{ with } |x_i - x_j| \leq 2 \ \forall \ i, j \in \{1, 2, \ldots, c\}\}. \tag{3}$$

Since this set is unaffected by any translation by any $(x, x, x, \ldots, x)$ (and no voter cares if his vote is translated by any such) we could also demand $x_1 + x_2 + ... + x_c = 0$, thus getting a $(c-1)$-*dimensional* polyhedral set P'. What is P (or P')?

When $c = 2$, we get a 1D interval as P' and we get plurality[14] voting. When $c = 3$, we get (as P') a 2D regular hexagon. When $c = 4$, we get (as P') a 3D "rhombic dodecahedron" equivalent (after a rotation and scaling) to the convex hull of the following 14 points: the 6 vertices $(\pm 2, 0, 0)$ of a regular octahedron, and the 8 vertices $(\pm 1, \pm 1, \pm 1)$ of the reciprocal cube.[15]

The *vertices* of P have the property that at least $c-1$ equalities $|x_i - x_j| = 2$ will hold. If we draw a c-node directed graph (directed arc $i \to j$ if $x_i - x_j = 2$) to represent such a vertex, then there are $\geq c-1$ directed arcs in the graph, with no two arcs consecutive (i.e., $i \to j \to k$ never happens, since that would force two coordinates to differ by 4). Hence the graph is bipartite, i.e. there are two kinds of vertices, "upper" and "lower" such that only vertices of *differing* kinds are joined by graph edges, and those edges are always directed from upper toward lower.

[12] In the limit V large, because of the central limit theorem. We shall assume the poll results are not reported in any greater detail.

[13] I.e., break or create a tie.

[14] Actually, the resulting system is not *exactly* the same as plurality. However, rational voters will always choose to make their vote lie at an interval *endpoint* in order to maximize its impact (unless they think the two candidates are exactly tied, in which case, presumably they would not vote at all). If all voters acted this way we would indeed have plurality voting.

[15] In general, P' is obtained by starting with a regular $(c-1)$-simplex, putting hyperplanes through each vertex perpendicular to the $(c-1)$ edges out of that vertex, and these $(c-1)c$ hyperplanes define the faces of P'; and (we shall soon see) the voting system obtained is always equivalent to range voting.

Then the upper layer in this bipartite graph must have all coordinates +1 (unless no arc comes out of that node; then any value in $[-1, +1]$ is ok). The "lower layer" must have all coordinates -1 (unless no arc enters that node; then any value in $[-1, +1]$ is ok). Each arc we add causes at least 1 node i to be forced to $x_i = +1$ or $x_i = -1$ (given that we *start* with 1 arc and 2 such nodes) so the total number of nodes which are either +1 or -1 must be: all c of them. Final *conclusion*: the vertices of P are precisely the 2^c hypercube vertices $(\pm 1, \pm 1, ..., \pm 1)$.

This may be projected onto the $(c-1)$-dimensional hyperplane $\sum_{i=1}^{c} x_i = 0$ to get P'. Thus to explain the somewhat mysterious claims a paragraph ago, note that a 3D cube viewed along a diagonal is a 2D regular hexagon, whereas a 4D cube viewed along a diagonal is a 3D rhombic dodecahedron.

Conclusion. We may regard this set P as the c-hypercube $|x_i| \le 1$, $i = 1, 2, \ldots, c$ (or regard P' as its projection along a diagonal into the $\sum_{i=1}^{c} x_i = 0$ plane).

Using $S = \{$the vertices of $P\}$ would cause us to adopt "approval voting." If the full set P (not just its 2^c vertices) is allowed, we get "range voting," which grants voters greater expressive freedom.

7 Rational voting in COAF systems

Before proceeding to consider under what conditions S is forced, uniquely, to be P, so that we are forced to get range voting... let us first understand precisely what rational voting, in COAF systems, generically is.

Definition. A "rational" voter votes in such a way as to maximize the expected utility of the election result, for him.

Thus (in the limit $V \to \infty$), his vote vector $\vec{x}$ is chosen to maximize[16]

$$\sum_a \sum_b (U_a - U_b)(x_a - x_b) P_{ab} \tag{4}$$

subject to $\vec{x} \in S$, where U_k is that voter's utility for "candidate k is elected," and P_{ab} is proportional to his perception of the probability that the election will be tied between a and b, so that his vote $\vec{x}$ will break that tie and thus have an effect. We are here modeling the probability densities as "smooth enough" that the probability of being "approximately tied with gap $\le G$" is proportional to G for small G, thus explaining the term $(x_a - x_b)$ in the sum.

Now, *generically,* all the P_{ab} are exponentially tiny as $V \to \infty$, and all their ratios are exponentially enormous. (All the P_{ab} can be approximately evaluated, to good enough precision for our purposes, with knowledge of the Gaussian distribution from the pre-election polls, by finding the L_2-norm of the min-norm point on the a-b decision surface, in the affined space A of §5; the P_{ab} will be decreasingly ordered in the same order as the *increasing* order of these norms. In the below we shall also assume, also "generically," that all the U_k are distinct.) Hence, in the limit $V \to \infty$ with $\vec{U}$ fixed, choosing the best $\vec{x}$ becomes trivial: always choose it, subject to $\vec{x} \in S$, to maximize $x_a - x_b$ for the a and the b with maximum P_{ab}. If this suffices to specify $\vec{x}$ uniquely, we are done. Otherwise, suppose this maximization has reduced S to the much smaller (but non-singleton) set S_1. Now consider the new a and b with the next largest value of P_{ab} and maximize $x_a - x_b$ subject to $\vec{x} \in S_1$. If the resulting set $S_2 \subset S_1$ is not a singleton, continue on in this way until $\vec{x}$ is uniquely determined; that is the rational vote.

It turns out that this algorithm for determining the rational vote simplifies drastically in the case when the Gaussian is spherically symmetric. We shall often **assume** this case. In that case, suppose the center of the Gaussian is $\vec{X}$. Suppose without loss of generality that the candidates are ordered by decreasing likelihood of election, e.g. candidate #1 is the frontrunner, with $X_1 = \max_k X_k$, and #c is in last place, $X_c = \min_k X_k$. Then the log-likelihood that candidate b will end up tied for the lead with some particular candidate a for some $a < b$, is proportional (in the $V \to \infty$ limit) to

$$\log P_{ab} \propto \frac{-1}{b-1} \sum_{k=1}^{b} (X_k - \overline{X}_b)^2 \tag{5}$$

where $\overline{X}_b \equiv \frac{1}{b} \sum_{j=1}^{b} X_j$.

This tells us that the rational vote $\vec{x}$ may be determined by considering the candidates k in decreasing order of X_k. Each time, we choose x_k to be *either*: the greatest or least possible value subject to $\vec{x} \in S$ and subject to the fact that we already chose x_j for $1 \le j < k$. (Because, generically, these $j \leftrightarrow k$ ties are enormously exponentially more likely than any ties involving candidates numbered$> k$: any choice not extremizing x_k, could not be utility maximizing.) Which? Since (by EQ 5) all ties between candidate k and candidates j are equally likely (for all $k-1$ possible choices of j, $1 \le j < k$) we want to choose x_k to maximize

$$\sum_{j=1}^{k-1} (U_j - U_k)(x_j - x_k). \tag{6}$$

Lemma 1 (Rational voting in COAF systems: Using "moving average" as threshhold maximizes utility) *Assume the candidates are ordered by decreasing likelihood of election according to the pre-election polls, and assume those polls yielded a spherically symmetric Gaussian. Let $\overline{U}_b = b^{-1} \sum_{i=1}^{b} U_i$ denote the average of the first b candidate utilities. Choosing x_k to maximize EQ 6, (thus choosing the rational vote-vector $\vec{x}$) is equivalent to the following:*

1. *If $U_k > \overline{U}_{k-1}$ then set x_k to the maximum allowable value.*
2. *Otherwise set x_k to the minimum allowable value.*

[16] This same expression is found on the bottom right of page 103 of [28], whose utility-theory formulation of elections is essentially the same as ours here.

Proof. EQ 6 may be rewritten

$$(k-1)U_k x_k - (k-1)x_k \overline{U}_{k-1} + \sum_{j=1}^{k-1} U_j x_j - U_k \sum_{j=1}^{k-1} x_j. \quad (7)$$

Of the four terms in EQ 7, the last two are constants unaffected by x_k. Thus if $k > 1$, maximizing EQ 7 is equivalent to maximizing

$$[U_k - \overline{U}_{k-1}]x_k \quad (8)$$

proving the claim. QED.

Thus, to determine (generically), your rational vote in a c-candidate Borda election, proceed as follows. First, award your favorite among the two poll-frontrunners c votes and the other 0 votes. Now, proceed through the remaining $c-2$ candidates k in decreasing order of their election likelihood (according to pre-election polls). If candidate k's utility exceeds the average utility of the previous $k-1$ candidates (whose votes we have already chosen) then award k the maximum available vote value (which, according to the rules of the Borda system, keeps decrementing starting from $c-1$), otherwise the minimum available vote value (which keeps incrementing starting from 0).

Thus a utility vector $\vec{U} = (3, 9, 1, 5, 0)$, with the candidates ordered in decreasing order of their rank in the polls, would translate into a Rational Borda vote $(0, 4, 1, 3, 2)$.

Finding the rational range vote is even simpler, because the maximum and minimum vote values are always just $+1$ and -1. Thus (generically) to find the rational range vote, award the higher-utility among the two frontrunners $+1$ votes and the worst -1 votes. Now, proceed through the remaining $c-2$ candidates $k = 3, \ldots, c$ in decreasing order of their election likelihood (according to pre-election polls) awarding $x_k = \pm 1$ if U_k is greater than, or less than, the average utility of the previous $k-1$ candidates.

We call this the *Moving Average Strategy.* Lemma 1 is related to

Lemma 2 (Average as threshhold maximizes utility in range voting) *Let* $\overline{U} = c^{-1}\sum_{i=1}^{c} U_i$ *denote the average of the* c *candidates's utilities. Choosing* $\vec{x}$ *to maximize*

$$\sum_{a=1}^{c}\sum_{b=1}^{c}(U_b - U_a)(x_b - x_a) \quad (9)$$

given that $|x_k| \le 1$ *for all* $k \in \{1, 2, \ldots, c\}$, *is equivalent to setting* $x_k = \operatorname{sign}(U_k - \overline{U})$.

Proof. Obviously, EQ 9 is maximized by making all coordinates of $\vec{x}$ be ± 1, and also obviously this should be done according to $x_k = \operatorname{sign}(U_k - T)$ for some threshhold T; the only question is what T is. To answer that, consider transferring some x_k from -1 to $+1$. The additive effect this will have on EQ 9 is

$$(c-1)U_k + \sum_{j \ne k} U_j \quad (10)$$

which is positive if and only if U_k is greater than the average utility $\overline{U}_{\ne k}$ among the $c-1$ other candidates. Since $c\overline{U} = (c-1)\overline{U}_{\ne k} + U_k$, this in turn happens if and only if $U_k > \overline{U}$. QED.

Essentially, lemma 2 says that the utility-maximizing range vote in situations without poll data, or in which the *poll data is a precise* c*-way tie* among the top c candidates (we assume any other candidates are exponentially enormously ignorable, and assume all the variances as well as all the means are equal, i.e. assume that the Gaussian is spherically symmetric) is pure ± 1's with the $+1$'s awarded to the above average and the -1's to the below average utility candidates. (More generally this is a *moving* average, but in the present special case it remains stationary.) This follows from EQ 4 since the P_{ab} are all equal – all ties are equally likely to arise in the election, and then to be broken by our vote. Thus rational range voting can also be understood even in non-generic (tied in the polls) situations.

It has, however, been left unresolved by lemmas 1 and 2 what the most rational vote is in the more complicated situation where the Gaussian from the pre-election polls is *not* spherically symmetric, i.e., where the $k \times k$ covariance matrix of the k vote coordinates is not just a scaled identity matrix. In that case the quadratic form in EQ 5 must be replaced by a different – but still positive definite – quadratic form and the P_{ab} no longer can be regarded as all equal in the c-way tie scenario of lemma 2.

8 Uniqueness of P

We now return to the main line of argument, starting from the end of §6.

Actually, to force rational voters to provide information, it was not necessary to make the face planes be $x_i - x_j = \pm 2$. One could have chosen $x_i - x_j = \kappa$ for various constants κ, and furthermore this could really only be piecewise constant. But if in addition we demand that S be *convex* then it is not possible for two different κ's to coexist for a fixed (i, j) because there is no convex set whose upper boundary contains both a nonzero measure chunk of a hyperplane, and a nonzero measure chunk of a different parallel hyperplane. Next, if we demand that S be *fair* (invariant under the $c!$ permutations of coordinates) then all the κ_{ij} must be equal to just *one* constant, which we may as well (by scale normalization) select to be $\kappa = 2$.

Let us recap. So far, we have demanded that our voting system be fair, convex, compact, additive, and have the property that rational voters can express independent information about at least $\lfloor c/2 \rfloor$ disjoint pairs of candidates, plus an extra half-disjoint pair if c is odd (or: alternative phrasing: "continuously variable information about at least $c-1$ candidates").

S is still not uniquely specified by these desiderata because, although we know that every hyperplane $x_i - x_j = \pm 2$ must determine a face, we have not forbidden other kinds of faces. However, there are several possible additional conditions, any one of which will cause S to become uniquely specified (and $S = P = \{\vec{x}$ such that $\forall i\ |x_i| \le 1\}$=hypercube):

1. We can demand S (when projected into the $\sum_{i=1}^{c} x_i = 0$ hyperplane to get S') have maximal $(c-1)$-volume (subject to our bound $\max_{i,j} |x_i - x_j| \le 2$ if $\vec{x} \in S$), thus granting voters the "maximum possible expressive freedom" subject to our previous desiderata. (Maximum volume immediately implies convexity, since S' must fill its convex hull, so S' must in fact then be the set $\{\vec{x} \in \mathbf{R}^c$ such that $|x_i - x_j| \le 2$ and $\sum_{i=1}^{c} x_i = 0\}$.)
2. We can demand that rational voters always have the freedom to give their favorite (i.e. any) candidate the largest possible number (i.e. +1) of votes (despite the fact that this favorite may have negligible chances of election according to the pre-election polls)[17].
3. We can demand that voters always have the option of being "honest" (albeit irrational) in the sense that they will always be allowed to do "scaled utility voting."

To summarize:

Theorem 3 (Uniqueness of range voting) *When the number of voters V goes to ∞ with the number c, $c \ge 2$, of candidates fixed and with each voter having a uniformly bounded utility for the election of each candidate, then: there is a unique*[18] *compact set $S \subset \mathbf{R}^c$, and a unique corresponding COAF voting system, obeying these 2 demands:*

1. *Voters (despite the constraint that they be rational in the presence of arbitrary generic pre-election poll results) can express continuously-variable independent information about at least $\lfloor c/2 \rfloor$ disjoint pairs of candidates, plus an extra half-disjoint pair if c is odd. (Or: alternative phrasing: "can express information about at least $c-1$ pairs of candidates.")*
2. *At least one of the following is true:*
   - (a) *S (when projected into the $\sum_{i=1}^{c} x_i = 0$ hyperplane) has maximal $(c-1)$-volume subject to $\max_{i,j} |x_i - x_j| \le 2$ if $\vec{x} \in S$.*
   - (b) *S is convex and each voter can choose to give an arbitrary candidate (i.e. his favorite) the maximum possible number of votes, even under the constraint that that voter must act rationally.*
   - (c) *Voters could, in principle, do honest scaled utility voting using the maximum possible scale, i.e. with a vote satisfying $\max_{i,j} |x_i - x_j| = 2$.*

The unique set S always is the hypercube $|x_i| \le 1$ and the voting system is range voting.

Proof summary. We found the rational voting strategy in COAF systems in §5 and §7. From this it followed from demand 1 as in §6 and §8 that the set S defining the COAF system had to be a c-dimensional hypercube $\{\vec{x} \in \mathbf{R}^c, \forall i\ |x_i| \le 1\}$ (or some equivalent projection of it) – but perhaps with additional faces. But the possibility of additional faces was shown in §8 to be ruled out by any of the assumptions of demand 2. QED.

Theorem 4 (Honesty in 3-way elections) *If $c = 3$ (3-way election) then (when $V \to \infty$) the unique COAF voting scheme such that*

1. *S is convex and compact*
2. *Has the property that either rational or honest voters will generically provide x_1, x_2, x_3 votes for candidates $1, 2, 3$ such that x_1, x_2, x_3 are ordered consistently with the voter's actual preferences (i.e. $x_1 \ge x_2 \ge x_3$ if the voter thinks candidate 1 is superior to 2 is superior to 3)*

is the cube $|x_1| \le 1, |x_2| \le 1, |x_3| \le 1$.

Proof summary. This really is just theorem 3 specialized to $c = 3$ and employing demand 2b.

It is educational to re-sketch the argument specially in this case. It ultimately traces to the fact that the 3-cube has a flat face *perpendicular* to the directions of voting for or against the two frontrunners, so it "costs the rational range voter nothing" (no decrease in expected utility) to move his vote toward his favorite candidate, even if that candidate is a nonfrontrunner – and indeed this is true even if he moves his vote maximally so that it becomes of the form $(\pm 1, \pm 1, \pm 1)$. Meanwhile, in Borda, Plurality, and Bullet (or any COAF system whose S lacks faces of these orientations), any attempt to grant your favorite candidate (if not one of the 2 frontrunners) your maximum vote, will force a decrease in your vote difference between the two frontrunners. This decrease is exponentially more likely, generically, to hurt you than the likelihood that your increased vote for your favorite will help you – so, rationally, you won't do it. QED.

In contrast to the above theorems:

1. In Borda, Plurality or Bullet, the rational voter can easily find himself with no way to provide *any* information about his most-loved candidate! For example, in a 3-candidate Borda=Dabagh vote, the rational voter will vote 2 for one of the two poll-frontrunners, 0 for the other frontrunner, and then, whether the remaining candidate is his most-loved or most-hated does not matter since he will have no way to express his feelings on the subject – he must award that candidate, by the rules of the Borda system, exactly 1 vote no matter what he thinks. If all voters are rational, one of the two poll frontrunners is then guaranteed to win unless there is an

[17] Ossipoff says voting systems that deny this ability have the FB, for "favorite betrayal," property. We discuss that in §9.2.

[18] Aside from scaling, and after projection into the $\sum_{i=1}^{c} x_i = 0$ hyperplane.

exact 3-way tie. With rational Plurality voting, the same is true. In rational Bullet voting, rational voters generically would always shoot down their least-liked among the 2 frontrunners, since a bullet for a non-frontrunner would, generically, be exponentially more likely to be "wasted," so the *non*-frontrunner would be guaranteed to win![19].

2. Range voting allows scaled utility voting but Borda, Plurality, Bullet, or Approval do not – so even if all voters wished to be irrationally honest (sacrificing strategy for the overall good of society) society couldn't gain the full benefits of their virtue, in those systems.

3. Rational Borda, Plurality, and Bullet voters will not be honest about their preference orderings in 3-way elections (and it is very common for elections to be 3-way, or effectively 3-way). Meanwhile theorem 4 says range voters will be.

In the light of Donald Saari's [32] love of the symmetry properties of the $c = 3$ Borda voting system (cf. §9.6), it perhaps is of interest to note:

Theorem 5 (Symmetry in 3-way elections) *In a 3-way election, range voting has the same symmetries as the Borda scheme.*

Proof: Both have sets S defining regular hexagons in the $x_1 + x_2 + x_3 = 0$ plane; it is just that the Borda and Range hexagons are rotated 30 degrees – dovetailed – with respect to one another. QED.

For more discussion of Borda see §9.6.

The net effect of theorems 3-5 seems to me to indicate that range voting is clearly superior to Borda and all other COAF schemes in ($c = 3$)-way elections having many well informed rational voters; and it also is arguably (§4) optimal for honest voters too, for any c.

9 Comparison with previous work

There has been a tremendous amount of previous work on voting systems [5][24][27][30][31][32][33]. Much of this work proceeds by defining a large set of properties that voting systems could have – and which plausibly sound like properties a *good* voting system should have – and then seeing which of them are obeyed by which voting systems.

Before I begin discussing voting system properties, let me avoid some possible confusion by mentioning an important point that many authors, unfortunately, do not. That is: we should consider each property in at least 3 scenarios:

1. Where the voters are "honest."

2. Where the voters are "rational" and their preferences are post-judged from their expressions of them as their votes. For many purposes this is effectively the same thing as the preceding – we are essentially *pretending* all the voters were honest! Therefore we shall usually merge this scenario with the preceding one.

3. Where the voters are "rational" but their preferences are judged from their true, privately known, utilities. Although these are empirically impossible to assess from the outside, if we are considering voters abstractly (or are generating artificial voters inside a computer simulation), then it is entirely possible (and desirable) to consider them.

9.1 *How range voting behaves with respect to Nurmi's list of voting system problems*

H.Nurmi organized much of his book [30] into chapters discussing one problem or another suffered by voting systems. Let us consider range voting's behavior on Nurmi's problems.

ch.4. Problem: many voting schemes can output non-transitive "cycles" of preferences. This is never a problem for either range voting or any other COAF system because real number $<$ is transitive. Also, if range voting is regarded as outputting only 1 winner, not a rank ordering, that is another reason it is no problem.

ch.5. Problem: Will a "Condorcet Winner" CW (who would win all pairwise head to head elections) always win? Answer: No (but, if CW were selected based on vote vectors, then Yes since $<$ is transitive for reals).

Proof: For *honest* scaled utility voters: If 51% of the voters think CW is highest utility, but only by a little, while 49% of the voters think CW is lowest utility, by a lot, then CW can (and should, for the good of society) lose in a (≥ 3)-candidate election.

For *rational* range voters: When the rational voter is deciding what to do using the procedure in §5, eventually he will mentally pair CW versus another candidate (or weighted average thereof), and hence CW will get a +1 vote in more than half the range votes, say 51% of them. But meanwhile, if the two frontrunners in the polls are A and B, and 73% of the voters think $U_A > U_B$, then A will get +1 in 73% of the range votes. In that case

[19] Therefore, notice that, with Bullet voting and rational voters, in a 3-candidate election, the pre-election polls would always be *wrong!* I call voting systems with this property (that rational voters will tend to act in such a way as to invalidate pre-election poll predictions) "suicidal." It is an interesting question (§11) to try to understand precisely which voting systems are suicidal and quantify by how much. It could be argued that much of the analysis in this paper – based on the assumption that rational voters will act as though they believe the polls are accurate reflections of what the other voters will do – would be invalid in suicidal voting systems. Then my arguments that, e.g., range voting is "best" among COAF voting systems, are undercut, and would need to be weakened to "best among non-suicidal COAF systems." To respond to that, I now argue that suicidal voting systems are, inherently, bad voting systems. This is because, in order for a system to *be* suicidal, it has to be highly manipulable by dishonest strategic voters, and highly unstable and vulnerable to rumors percolating among the rational voters and affecting their strategic decisions. Therefore, range voting is presumably better (in some sense I am admittedly leaving vague and intuitive) than suicidal COAF voting systems, also, and so, really, my advocacy of range voting is *not* undercut.

A would win the election and CW would not. (This is probably a bad thing.) QED.

Beware of confusion. There are really 3 cases to consider here, corresponding to the three items in the numbered list of §9.

A. We could consider finding a Condorcet winner by using the *same* voting system (in our case range voting), using the *same* votes (but ignoring all coordinates in all vote vectors except for coordinates i and j) in the pairwise i versus j election. With that definition, Range voting will always elect a Condorcet winner regardless of whether the voters are honest.

B. But if the pairwise i versus j elections are conducted using a *different* voting system and votes, namely plurality, then Range voting need *not* elect a Condorcet winner.

C. Finally, if Condorcet's pairwise i versus j elections are conducted either using plurality based on the private mental utility values in the voter's heads, or using true-utility summation, or using range voting re-votes, then, for *rational* range voters, range voting need *not* elect a Condorcet winner.

ch.6. Monotonicity? If a voter changes a preference relation $U_A < U_B$ to $U_A > U_B$, will that increase (anyway never decrease) the probability A wins? And will it always increase (or at least never decrease) the probability B loses? Yes, this is true for range voting with either rational (generically), or honest voters[20].

ch.7. Unanimity? If all voters agree on some ordering of the candidates, will that be the ordering output by the voting system? If the voters votee honestly, yes. For range voting with rational voters and $c \leq 3$ candidates, yes (though conceivably the order would be ambiguous due to a tie). But with $c \geq 4$ and rational range voters, not necessarily: If each voter thinks the 4 candidate utilities are $U_A = 0, U_B = 9, U_C = 100, U_D = 11$ where the candidates are ordered by decreasing likelihood of election according to the pre-election polls, then rational range voters will vote $A = -1, B = +1, C = +1, D = -1$, mis-ordering $U_D > U_B$.

Here I've assumed that the rational voters are using the Moving Average Strategy of §7, and I'll continue to assume this when constructing similar examples throughout this section.

But even when $c \geq 4$ and with adversarially chosen pre-election poll data, the correct *winner* will always be obtained by a range vote (by either rational or honest voters) if there is a unanimous consensus that candidate is best. (Though conceivably it will be a tie.) This is despite possibly wrongly ordering the losers. (For rational voters, Plurality and Borda do not have this unanimous-winner property, but Bullet, in the absence of ties, does.)

ch.9. Problem: how to encourage voter honesty (many voting schemes discourage it)? Well, of course, range voting is exactly designed to be best possible in this respect among a wide class of voting systems. Despite that, it is not perfect, as is shown by the 4-candidate election example above in which rational voters will, in their votes, dishonestly indicate their ordering of two candidates. But when $c \leq 3$, rational range voters will be completely honest about their relative preferences. Indeed (this idea is due to Brams and Fishburn [6]) if each voter's candidate-utility vector $\vec{U}$ is "trichotomous" (i.e. each U_i is in some voter-dependent 3-element set) then each rational range voter will always produce a vote $\vec{x}$ compatible with his ordering of the candidate utilities U_i.

Gibbard's dishonesty theorem: It is not surprising that rational voters will sometimes be dishonest about their preference orderings in (≥ 4)-candidate range voting. This is because Gibbard [21][35] showed that rational voters *must* sometimes be dishonest in *any* deterministic non-dictatorial voting scheme[21] with ≥ 3 possible outcomes. Gibbard then went on to show that *non-deterministic* voting systems were possible in which rational voters would always be honest about their preference orderings, indeed for every possible value of c, the number of possible election outcomes. Unfortunately (Gibbard showed) there are exactly two such systems (and probabilistic mixtures of them of the form "use system #1 with probability p, otherwise use system #2"), and neither is very attractive:

1. (Random dictator) Your vote is the name of one candidate. One out of all the V votes is selected at random as the winner.
2. (Majority choice on random pair) Your vote is a permutation ordering all c candidates. Two of the c candidates are then selected at random and the winner is the more preferred among these two candidates.

Unfortunately, Gibbard said, "clearly" these systems are unacceptable in practice "because they leave too much to chance." (The ith of these two systems is easily capable of electing the ith-worst candidate, who might have unboundedly worse utility than any candidate who could be elected by a deterministic procedure. See also our Monte Carlo experiments in §10.)

Why Gibbard doesn't conflict with theorem 4: Gibbard's theorem seems to contradict our theorem 4. There are two reasons this is not actually a paradox. First, range voters do *not* provide 3-permutations as their votes (as Gibbard demanded in his definition, which we now see was too restrictive, of "voting scheme") – they

[20] However (David Pennock remarks) If a new candidate C is added, even if everybody hates C, then the results between A and B might change (for rational range voters influenced by adversarially chosen poll data depicting C as a likely winner). Such an extraneous candidate cannot affect elections with *honest* range voters, however. But for *Borda* with honest voters, the addition of an extra candidate who has no chance of winning, can cause the overall scores of all the original candidates to *reverse* their order [5]!

[21] A *voting scheme* to Gibbard, is a map from preference orderings among the candidates, provided by voters, to the identity of the winner. It is *dictatorial* if the winner's identity always agrees with the top preference of one particular voter, called a dictator.

provide 3 real numbers. Second, for rational voters, theorem 4 says *some* ordering of these three numbers using "$\leq$" will be compatible with the true ordering (using "$<$") of this 3-permutation; this is a slightly weakened form of "honesty."

9.2 Some other properties voting systems may have

Mike Ossipoff made a list on his web page of his favorite properties of voting systems. Ossipoff calls range voting the "Olympic 0-10 system" since it is the system (based on the interval $[0, 10]$ rather than $[-1, +1]$; that makes no difference) used by judges of figure skaters and gymnasts in the Olympics. Let us consider range voting's behavior for Ossipoff's property list.

FB: Favorite betrayal: Does the voting scheme sometimes force a rational voter to give his favorite candidate less than the maximum vote? No: Rational range voters will not betray their favorites.

ML: Majority loser: If 51% of voters rank a candidate last, can he win? Related is: **Condorcet loser:** Can a candidate who would lose every pairwise election, win?

Ossipoff feels that neither ML or CL are very important in practice, since they pertain to unlikely embarrassments. For the following voting systems the answers to both questions are the same:

Range (rational voters): No[22].

Range (honest voters): Yes. (However, if CL were selected based on ignoring all coordinates of all vote vectors, except for one coordinate pair, to determine the pairwise election losers, then summing, then No since $<$ is transitive for reals. Cf. our discussion of Nurmi's ch. 5 in our §9.1.)

Example proving this: Suppose 51% of voters think ML is lowest utility, by a little; 49% think he is highest utility, by a lot. This example is very important because it demonstrates that the **ML** and **CL** criteria are poor ones – in this example the Majority Loser *should* win, for the overall good (i.e. summed utility) of society. QED.

UD: Unanimous domination: If it is unanimously agreed that $U_A < U_B$, can A still win? Answers: Range voting with honest voters: No. Range voting with rational voters: Yes (but not in (≤ 3)-candidate elections, unless there is a tie).

Proof: Let the candidates, in decreasing order of election likelihood according to the pre-election polls, be A, B, C, D, E, F. Suppose 25% of the voters think $U_A = +10, U_B = +11, U_D = -10, U_F = -10$, 25% think $U_A = -10, U_B = -11, U_D = +10, U_F = +10$, 25% think $U_A = +2, U_B = +1.99, U_D = -15, U_F = -15$, and 25% think $U_A = +1.99, U_B = +2, U_D = -15, U_F = -15$ and all voters agree $U_C = +1 > U_E = -1$. Then the 4 respective kinds of voters will vote $(-, +, -, -, -, -)$, $(+, -, +, +, +, +)$, $(+, -, -, -, +, -)$, $(-, +, -, -, +, -)$ for (A, B, C, D, E, F) and hence the election will be won by E, despite the unanimous agreement that $U_C > U_E$. (For the validity of **UD** for (≤ 3)-candidate elections, see theorem 4.) QED.

9.3 More properties

I'll now list some other properties voting systems can have. In the Ossipoff list in §9, voting systems "passed" a criterion (such as **FB**) if their answer to the question was "no," otherwise they "failed." But in the yes/no questions in the list below that is reversed: pass=yes, fail=no.

Wk: Work: How much work does it take to administer an election? For additive schemes, the work is small, since each voting site only needs to transmit the sum of its vote vectors (c numbers) to the central agency. For the Condorcet Least Reversal system, the work is larger, since $\binom{c}{2}$ numbers (pairwise comparison vote counts) may need to be transmitted and stored. For the STV systems, the work can be *far* larger, since apparently either $c!$ counts (one for each c-permutation) need to be transmitted to (and stored at) the central computing agency, or all V votes (Vc numbers) need to be transmitted and stored. (With $c = 12$ and $V = 10^8$, $Vc = 12 \times 10^8$ and $c! \approx 4.8 \times 10^8$.) In my opinion, *this is often so severe an indictment of STV systems that it removes them from contention with additive systems.*

H3: Honesty in 3-way elections: Will voters choose to be honest about their preference orderings, when voting?

MI: Maximum information: Will voters provide "maximum information" (i.e. provide their opinion of the maximum possible number of candidates) in their vote? [Only range voting really passes this test, cf. theorem 3, but the systems allowing c-permutations as votes arguably permit more information in votes than systems such as Bullet.]

Mo: Monotonicity: Is it true that if a voter changes a preference relation $U_A < U_B$ to $U_A > U_B$, that cannot decrease the probability A wins, and cannot decrease the probability B loses?

IV: Incentive to vote honestly: Is it true that your participation as an honest voter cannot decrease the expected utility (by your utility measure) of the election outcome (versus not voting at all)?

[The failure of **IV** has also been called the "no-show paradox." **IV** and **Mo** are both always true for additive systems. But they both are violated in Hare-STV voting [7][12][5] and Condorcet Least Reversal voting.]

Mo and **IV** may be very important properties for a voting system to have, since without them, people may be discouraged from participating as voters at all.

SU: Scaled Utility voting: Is scaled utility voting possible? Are there continuous degrees of freedom allowing honest voters to express continuous changes in their perceived utilities for the candidates?

UW: Unanimous winner: Does a unanimous consensus winner (whom all voters agree, has maximal utility) always win?

[22] **Proof:** Mr. Loser will get a -1 vote more than 50% of the time; the winner (in the Moving Average Strategy) won't. QED.

CN: Consistency: A voting system is "consistent" if when the electorate is divided arbitrarily into two parts and separate elections in each part result in the same candidate's election, then an election of the entire electorate also elects him. (If a voting system is not consistent than it is especially subject to manipulation by strategically configuring election districts. All additive systems are automatically consistent – at least if we ignore considerations of different strategic decisions being made if the voters knew their votes were not going to be used in a larger election but only in a smaller election, and if the poll results they based their strategies differed for the two sub-electorates.)
CW: Condorcet winner: Will the Condorcet winner (who would win any pairwise election) win the election?

[Note: in any additive scheme if the CW were judged from the votes and the same voting scheme were used in the pairwise sub-elections, this is true – although for range voting if the pairwise elections were run using *plurality*, **CW** is false. Cf. my caution warning in my discussion of Nurmi's ch. 5 in our §9.1. So to make things interesting we'll instead judge the CW from the voters's private mental utility values using plurality voting to do the pairwise elections. Then as I've said I do not agree voting systems "should" always "pass" the CW test, since the best summed-utility candidate is fully capable of not being CW, cf. end of §10.7.]

CN and **CW** may be important because, if a voting system ever suffered an obvious consistency failure, or if it ever failed to elect a Condorcet winner, then there would probably be a tremendous public outcry to overthrow the ("obviously flawed") voting system and punish its creators. Thus the failure of **CN** and **CW** in STV could be a serious indictment of STV. Fortunately additive systems, including range voting, always obey **CN** and **CW** (as judged from the votes, that is).

9.4 Table summarizing properties of 15 voting systems

Here is how 15 of the voting systems mentioned in this paper act with respect to these properties.

Key to the table: Properties are listed roughly in decreasing order of how important I think they are in practice. For **Wk**, **MI**: `L`, `M` , `H`, denote low, medium, and high work and information/vote. For other columns: `P`, `F`, and `.` denote "Pass," "Fail," and "inapplicable." `P-`: worse than `P` since passing this test is only possible when ties are broken in unlikely ways. `?`: Unknown, insufficiently well defined, and/or conjectural.
`*`: In these two voting systems, winner is partly chosen by random chance.

voting system	Wk	FB	H3	MI	Mo	SU	IV	UD	CW	UW	ML	CN
RaH	L	P	P	H	P	P	P	P	F	P	F	P
RaR	L	P	P	H	P	.	.	F	F	P	P	P
CLRH	M	P	P	M	F	F	F	P	P	P	P	F
CLRR	M	F?	F	M?	F?	F	.	P?	F?	P?	?	F?
STVH	H	P	P	M	F	F	F	P	F	P	P	F
STVR	H	F?	F	M?	F?	F	.	P?	F?	P?	P?	F?
PlH	L	P	P	L	P	F	P	P	F	P	F	P
PlR	L	F	F	L	P	F	.	F	F	F	P	P
BoH	L	P	P	M	P	F	P	P	F	P	F	P
BoR	L	F	F	M?	P	F	.	F	F	F	F	P
BuH	L	P	P	L	P	F	P	F	F	P-	P	P
BuR	L	P	F	L	P	F	.	F	F	P-	F	P
RaDi*	L	P	P	.	P	.	P	P	F	P	F	F
RaPM*	L	P	P	.	P	.	P	P	P	P	P	F
CopeH	M	P	P	M	F	F	F	P	P	P	P	F

(However, I do not agree that the CW and ML tests *should* be "passed;" illustrative example of why in §9.1.)
Voting systems (definitions in §3, 3.1, 9.1):

```
RaH  Range(Honest)
RaR  Range(Rational)
CLRH (Condorcet's) Least-Reversal(Honest)
CLRR (Condorcet's) Least-Reversal(Rational)
STVH Hare Single-Transferable-Vote(Honest)
STVR Hare Single-Transferable-Vote(Rational)
PlH  Plurality(Honest)
PlR  Plurality(Rational)
BoH  Borda(Honest)
BoR  Borda(Rational)
BuH  Bullet(Honest)
BuR  Bullet(Rational)
RaDi (Gibbard's) Random Dictator
RaPM (Gibbard's) Random Pair Majority
CopeH Copeland(Honest)
```

I now sketch proofs for some of the entries (all the difficult ones) in this table which have not already been proven.

CLRH is not monotonic: Suppose you think $U_A > U_B$. Then your vote can cause the number of reversals needed for A to win (if you think $U_X, U_Y > U_A$ and so does society) to increase by more than the number of reversals needed for B to win (if society says $U_B > U_X, U_Y$; this is entirely possible since the digraph can have nontransitive cycles). QED.

CLRH is not consistent: Suppose society 1 says A loses since the societal judgement is $X > A$ with margin 10 votes, while meanwhile B wins (since B needs only 1 vote reversed). Suppose society 2 says A loses since the societal judgement is $Y > A$ with margin 10 votes. Suppose B also wins (since B needs only 1 vote reversed) in society 2. Then in the disjoint union of society 1 and 2, it is entirely possible that the combined societal judgement is $A > X$ and $A > Y$ with margin, say, 17 votes each, so that A is the winner, requiring zero vote reversals, in the combined society (whereas B still requires reversing 2 votes).

CLRH also disobeys **IV** for the same reason. Similar examples show Copeland disobeys **IV**, **Mo**, and **CN**.

STVH: The Majority Loser **ML** cannot win. Proof: Suppose he can win. Then eventually it gets down to 1 opponent versus ML, after everybody else has been eliminated; then ML loses. QED.

STVH: Condorcet winner **CW** can lose: Suppose CW is 2nd ranked by all voters, with the other $c-1$ candidates splitting the 1st rankings evenly. Then CW is eliminated in round #1. QED.

STVH: Can violate **CN**. Example: In voter subset I, let there be 70 votes of the form $A > B > C > D$, 60 of form $B > A > C > D$, 53 of form $C > B > A > D$, and 9 of form $D > C > B > A$. In voter subset II, let the votes be 8: $A > D > C > B$ and 5: $D > B > A > C$. Within each of these voter subsets, A wins a Hare-STV election, but the combined election is won by B. QED.

PlR: **UW** (and hence **CW**) can and will lose if he is not one of the two frontrunners, since it is not rational to vote for anybody besides one of the two frontrunners (generically). QED.

STVH: **UD** is valid: If all voters agree $U_A > U_B$, then B will be eliminated since not top ranked by anybody. QED.

PlH: **UD** is valid: If all voters agree $U_A > U_B$, then B cannot win since gets no votes. QED.

BoH: **CW** can lose if 49% of voters think CW is bottom ranked, 51% think top ranked, and meanwhile some other candidate is always in 1st or 2nd place in all voter rankings and hence wins. QED.

BuH: **CW** can lose: for same reason as BoH.

H3 is false for Condorcet Least Reversal: The fact that voting dishonestly is sometimes strategically wise in 3-candidate elections, is a consequence of Gibbard's theorem (see my discussion of ch.9 in §9.1). A typical specific example is shown in figure 1, where it pays to dishonestly rank the opposing frontrunner artificially "last."

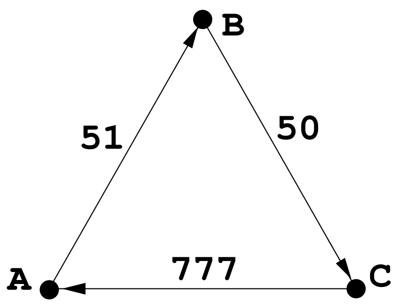

Figure 1: Suppose the pairwise margins of the rest of society's votes ($\rightarrow$ points to more preferred) are as shown. Then B would win a Condorcet Least Reversal election (the 50 would be reversed). Now suppose you prefer C over B over A. If you say so honestly in your vote and add that vote to the rest of society, then we get 52, 51, 776 and your vote has no effect. If you dishonestly pretend to prefer A over C over B, then C wins (margins: 50, 51, 778). Thus, in the Condorcet Least Reversal system, voting for your favorite actually can cause him to lose!

H3 is false for Hare-STV: This again is a consequence of Gibbard's theorem. But Joe Dee worried me by saying: "[For 3-candidate Hare-STV] Consider the voter who prefers A over B over C. There seems to be no harm to that voter under STV if he votes honestly because... if A gets eliminated the vote goes to B."

If it is assumed that every B supporter has A as his second choice, and every A supporter voter has B as second choice, then Dee is right: there is no penalty for honesty. However, if there are also many voters who prefer B to C to A (and say so in their votes), then by being honest our voter risks splitting the A-B vote, causing B to be eliminated in round 1. Then in round 2, *not* all the B votes will be transferred to A – some will instead go to C – in which case, A could then lose the second round and C would win.

Numerical example: suppose the votes among the 348 other voters are $C > A > B$: 150; $B > C > A$: 50; $A > B > C$: 99; $B > A > C$: 49. In this case our voter's 1 additional $A > B > C$ honest vote would cause B to be eliminated in round 1, at which point C would beat A in the next and final round by 200 to 149. However, if our voter had dishonestly voted $B > C > A$, then A would have been eliminated in round 1, at which point B would win the final round versus C, 199 to 150. In this case, our voter's honest C-last vote actually caused C to win[23]! In this example, the alternative dishonest vote $B > A > C$ also works; thus here again, artificially ranking the most-disliked (C) of the two frontrunners (B, C) artificially "last" is a best strategy.

Conclusion: it is not always strategically best to be honest in one's 3-candidate Hare-STV, or Condorcet-LR vote. That is also true for any other ranking-as-vote system, by Gibbard's theorem; *but* it *does* make sense to be honest, always, in 3-candidate *range voting.*

9.5 Arrow's impossibility theorem and its ilk

K.J.Arrow won the Nobel prize in economics substantially for his elucidation of "Arrow's impossibility theorem" [20]. This theorem, in a form stated and proven by P.Fishburn [16], is as follows.

Theorem 6 (Arrow) *Consider the following assumptions about a voting system intended for V voters considering c candidates.*
A1: $V < \infty$.
A2: $c \geq 3$.
A3: All candidate preference orderings are admissible for each voter.
A4: If all voters agree $U_A > U_B$, then the output of the voting scheme will be an ordering of the candidates in which $A > B$.
A5: Let there be two sets of V voters each. Suppose the ith voter in set 1 has the same relative ranking of A versus B as ith voter in set 2, for all i. Then the voting scheme on set 1 will come to same conclusion about $A < B$ or $A > B$ as the voting scheme on set 2.

[23] In this example, the Condorcet digraph again exhibits a nontransitive cycle of preferences; reversing the least-margin preference (B-C) according to Condorcet's prescription, would result in a C victory (with A second). But Borda would have resulted in a tie for the lead between A and B 497 to 497 (vs. 350 for C) – and our extra voter's vote would break this tie.

A6: No dictator (a voter such that if he says $U_A > U_B$ then the voting scheme will conclude $A > B$, for all A, B) exists.
Then: A1-A6 are inconsistent.

Remark: Fishburn also showed A2-A6 are consistent, but Kirman & Sondermann [25] argued that the voting schemes embodying A2-A6 with an infinite number of voters would contain "effective" dictators.

Assumptions A1-A6 seem to be extremely reasonable properties to demand of a voting system, so Arrow's theorem seems extremely depressing. How, then, can I be making claims that range voting is an excellent voting system, if there cannot be an excellent voting system? What is the relationship between Arrow's theorem and range voting?

First, let us criticize the underlying model of a "voting system." Arrow and friends all employ a model which is a map $R_1 \times R_2 \times R_3 \times \cdots \times R_V \rightarrow R$ where R_i is a c-permutation encoding the preferences of voter i, $i = 1 \dots V$, and R is another rank ordering, namely the "election result." Then they show no map satisfying A1-A6 can exist.

Range voting does *not* need to (and in my point of view in the present paper, does not) live inside this model. It does not output a permutation; it is only interested in (and the voters are only interested in) finding the 1 winner. Nobody cares about ordering the losers[24]. In that case A4 need not apply.

This "output format" objection is somewhat overcome in a different version of Arrow's theorem by A.Gibbard [22]. Gibbard imagines a voting scheme which outputs a set W of "winners" and inside which (but not necessarily output) there is a social preference ordering relation ">" which may, however, be rather weak, e.g. it is entirely possible that neither $x > y$ nor $y > x$ hold.

Theorem 7 (Improved Arrow) *Consider a voting system which inputs V preference permutations and outputs a nonempty strict subset W of the c candidates (the "winner" subset). Consider these axioms.*
$A1'$: $V < \infty$.
$A2'$: $c \geq 4$.
$A3'$: All candidate preference orderings are admissible for each voter.
$A4'_1$: If every voter thinks $U_B < U_A$, then the social preference includes the relation "$A > B$.".
$A4'_2$: If the social preference includes the relation "$A > B$" then $B \notin W$. $A5'$: Let there be two sets of V voters each. Suppose the ith voter in set 1 has the same relative rankings within some subset S of the candidates as does the ith voter in set 2, for all i. Then the voting scheme on set 1 will generate the same social preference $>$ relations between S-elements as the voting scheme on set 2.
$A6'$: No voter has "veto power," allowing him, for any A,B, to singlehandedly prevent the social preference $>$ relation from including "$A > B$." Then: $A1'$-$A6'$ are inconsistent.

But, despite this change of the output from an ordering to a winner subset, the *input* of the voting system remains wrong. In *my* model, the ultimate input to the system is *not* c-permutations; instead there is more: actual numerical utility values.

Since several utility vectors can be compatible with the *same* preference ordering, it is easily possible[25] to construct examples of two sets of voters violating A5 (and $A5'$) in which, for the overall good (utility-sum) of society, the winners really *should* differ. Thus, with *utilities* as input, it is certain that demanding A5 or $A5'$ is not always desirable; it actually causes societal harm. To defeat this objection, I suppose one might try to prove a new impossibility theorem using a reformulation of A5 based on utilities, e.g.:

$A5''$: Let there be two sets of V voters each. Suppose the ith voter in set 1 has the same utility values U_A and U_B for two candidates A,B as does the ith voter in set 2, for all i. Then the voting scheme on set 1 will come to same conclusion about $A < B$ or $A > B$ as the voting scheme on set 2.

But: no such impossibility theorem based on $A5''$ can hold because honest (unscaled) utility voting (§4) in fact *does* satisfy A1-A4, $A5''$, and A6.

Another criticism of A5 is: in my model, in fact strategic voters will consider their actual utility values and multiply them by probabilities of various election results (their analysis vastly simplifies when $V \rightarrow \infty$ since these

[24] S.J.Brams has complained that *he* cares about loser-performances because

1. Losers come back to win subsequent elections, because they performed well enough in earlier ones.
2. They inform winners about the preferences of their nonsupporters and may influence which way he should "lean" to appease them.

For these reasons Brams feels motivated to vote dishonestly even in a 3-candidate range election. My response is that indeed *everything* becomes mathematically disgusting when one is considering elections in which loser-performances affect utilities. For example, in the Bush-Gore-Nader 2000 election, I wanted 3rd-place candidate Nader to get $> 5\%$ of the votes, which would have assured him of getting millions of dollars in retroactive campaign financing. Thus I cared about this particular loser-performance. In my state of New Jersey, Gore was well ahead of Bush in the polls, so my vote for Gore or Bush was extremely unlikely to have an effect, say probability$\approx 10^{-200}$. Meanwhile Nader was going to get only 2-3% of the vote (said the polls) so it was *also* extremely unlikely Nader could reach 5% to get the money – say 10^{-100}. But even if money-for-Nader was an event 10^6 times less important to me than deciding the election winner, the fact that $10^{-100} \gg 10^{-194}$ indicates that my most rational vote would be for Nader! The point is that the combination of

1. generically exponentially tiny event probabilities,
2. the presence of payoffs based on non-winner results as well as just the winner (even very much smaller payoffs!)

will lead to incredibly cockeyed random-seeming thinking among strategic voters in all COAF systems, and will destroy the simplicity of rational-voter thinking in those systems that had arisen from (1) alone (§5). By assuming (as I do throughout this paper) that *only* the winner matters, such insanity is avoided.

[25] Example. Let there be two sets of two voters. The utility 3-vectors for the 3 candidates on voter-set 1, are $(0, 1, 2)$ and $(0, 3, 1)$ and on voter set 2, are $(0, 1, 3)$ and $(0, 2, 1)$.

probabilities go exponential) when deciding how to vote. Thus they genuinely will change their voting decisions even without *any* change in their preference permutations, just changing their utility values. Arrowists do not conceive of that possibility.

I think this willful ignorance has a lot to do with why Arrow gets into problems with "nontransitive preference cycles." Also, since I assume strategic voters are aware of pre-election poll statistics, my model has both different input (voters's mental utilities *and* the pre-election poll mean and covariance data) and different output (name of 1 winner), than Arrow's model of a voting system in which inputs are actual votes, and the output is a rank ordering.

Strategic voters certainly can and do look at the pre-election poll data and consider the status of other candidates besides A and B, when they are deciding how to vote about A and B. Thus it is foolish to assume (Arrow's A5) that they do not, can not, or should not. A5 is totally foreign to my picture where I consider private mental utilities as the true input to the system, not actual votes.

So the real problem with these theorems seems to be assumption A5 (or A5′).

Summary: Hopefully this has cleared the smoke from the air about Arrow's theorem. One can and should dismiss Arrow's assumption A5 (or A5′) and the model in which the input is preference permutations instead of utilities. (Note: honest range voting satisfies all of A1′-A6′ *excluding* A5′.)[26] It then *is* possible to find adequate voting systems, and among all such voting systems (or some large subclass of them, such as COAF systems), we can and should try to define and find the "best" one – the goal of this paper.

9.6 *Saari's championing of Borda count voting*

Donald Saari has championed "Borda count" voting [31][32], the COAF system in which S is the $c!$ permutations of $(0, 1, 2, \ldots, c-1)$. But I disagree. I feel that Borda is clearly not as good a system as range voting for either honest voters or rational voters. Honest voters would like a system permitting scaled-utility voting. Rational voters would be severely frustrated by their inability to elect, via a 3-candidate Borda count vote, the candidate unanamously top-ranked by all voters, unless he is one of the two frontrunners in pre-election polls[27].

Two of the reasons Saari likes Borda voting are: (1) He considered the class of Weighted Positional Voting (WPV) systems. He showed *Borda (with honest voters) is the only fair WPV system also satisfying "reversal symmetry*[28]*."* Saari then considered the "dictionary" mapping the voters's candidate preference orders (i.e. V permutations of the c candidates) into the c-permutation output by the voting system. (2) The more entries such a dictionary could have, the more Saari considered a voting system "paradoxical." Saari showed that *Borda is the uniquely least paradoxical WPV system.*

Let me counterargue.

1. I've worked with COAF voting systems, a highly general class. Because WPV systems are merely an infinitesimally tiny subclass of COAF systems (namely the ones with S being the permutations of one fixed vector $(W_1, W_2, \ldots, W_c)$ of constant "positional weights"), there is no reason to care about optimizing over them if we can instead optimize over COAF.

2. In particular, honest range voting *also* obeys fairness and reversal symmetry. This would not be possible (by Saari's theorem) if honest range voting were a WPV system – but we evade Saari's theorem by working in the wider class of COAF systems (cf. theorem 5).

3. It is wrong to model the input to the voting system as being V preference permutations. Really, the true input is V real utility c-vectors. Saari, and every WPV system, ignore (and prevent the voter from honestly expressing) the fact that a voter cares more about making A beat B if $U_A - U_B = 999$, than he cares about making B beat C if $U_B - U_C = 0.01$.

4. Nobody cares about rank-ordering the losers! We care about finding the winner. (In fact, I do not know of any use for a voting system outputting a full ordering.) So Saari's "dictionary" is dominated by irrelevancy.

5. Saari ignores the reality that voters are rational – instead modeling them as imbeciles who always vote "honestly," no matter how tactically stupid that is.

Saari's paradox theorem is beautiful, but do not be deluded into thinking it tells us much about how to build a good voting system. It doesn't.

9.7 *Other work – computational complexity*

Our point has been that range voters have both incentive and opportunity to provide a lot of honest information in a range vote. Bartholdi and Orlin [1] had the idea that a different way to prevent voters from being dishonest-rational would be if the *computational complexity* of determining how to be a rational voter were so high, that voters would simply give up on trying to figure out "tactics" and (as a last resort) simply vote honestly!

[26] If the voters are also assumed to have access to pre-election poll results which they can use to vote "strategically," then A4 also needs to be dismissed and the model needs to be modified even further. Also, I want to make it clear I am not claiming range voting with rational-dishonest voters satisfies A5″ – it doesn't.

[27] Admittedly, such a candidate would probably be high ranked in the polls. I am simply trying to dramatize the fact that, in 3-candidate Borda elections, the two candidates with the most advertising and loudest propaganda have essentially 100% chance of being elected purely because this loudness causes rational voters to *think* they have a high chance of being elected, totally independent of those two candidates's actual or perceived virtues.

[28] For another characterization of Borda, see [36].

They proved that it is NP-hard [19] to determine how to change your vote in the "single transferable vote" system in order to change the winner.

This was a cute idea, but: do not be deluded into thinking such results have anything to do with building a good voting system. This is because none of these NP-hardness results hold in the limit $V \to \infty$ (large number of voters) with the number c of candidates held fixed (or, more generally, with $c = O(\log V)$). Indeed, in this limit (which is, apparently the one relevant for elections in which humans participate) these computational tasks are easy, i.e. linear or even very sublinear time (or, more generally, in P). Also, even without a bound on c, it is usually still easy to think of a dishonest vote which seems to be more utilitarian than the true honest vote, despite difficulty in finding the *optimal* way to be dishonest. Thus, dishonesty is in no way prevented or discouraged.

These arguments apply with special force to rebut similar criticisms by Bartholdi et al. [2] of various voting systems for which it is NP-complete to determine *the winner*! For example, they showed NP-hardness for a version [37] of Condorcet least reversal voting in which the least number of vote reversals possible were employed so that the graph of pairwise elections would become acyclic. Again, do not be deluded into thinking this matters, because the elections in human history all have had a small enough number of candidates that deciding the winner is easy, so this is no reason to rule out these voting systems.

Incidentally, the least-reversal Condorcet scheme, defined in §3.1 and studied in my Monte Carlo experiments in §10, involves only determining *one* winner, with no attempt to find an acyclic ranking of *all* the candidates, and it is trivially in P to find the minimum number of vote pair-preference relation reversals needed to accomplish *that* goal. My form of Least Reversal voting, however, obviously *is* NP-complete (which was not mentioned by [2]) if it is used to choose w out of c "winners." That is because finding the required min-weight arc set is NP-complete (although with $w = 1$ it is always linear time) because of an easy reduction from GRAPH PARTITIONING [19]. However, for elections with $c \leq 30$ candidates, this is no obstacle since it is feasible to consider all $\binom{c}{w}$ node-subsets exhaustively.

10 Monte-Carlo experimental comparison of different voting systems

There is an extremely simple and obviously best way to compare voting systems. We simply construct, inside our computer, V artificial voters and c artificial candidates. By some randomized algorithm each voter assigns a utility to each candidate. (For the methods I used for assigning utilities, see §10.2.)

Now, we perform an election. Since, in our artificial election (unlike real elections) we *know* all the true utilities, we know exactly how much utility loss (summed over all voters), society suffered, for any given voting system, versus the hypothetical election of the true, maximum summed utility candidate. Call that utility difference, the "regret" of that voting system.

Definition. The "Bayesian regret" of a voting system is the (nonnegative) expected difference between the utility (summed over all voters) of the election winner that system produces, versus the maximum-possible (summed) utility which would have resulted had the best candidate always won.

We can perform a million simulated elections to compute the expected regret, to high accuracy, for each voting system we program. These regrets provide a simple way to compare voting systems.

10.1 Related previous work

This idea (of using Bayesian regret as a quality measure for voting systems, and using a computer Monte Carlo study to evaluate the regrets) was not first invented by me. It is present, e.g., in Bordley's 1983 study [4] and is there attributed, at least in part, to J.C.Harsanyi in 1955. Some previous computer experiments [27] unfortunately used the "Condorcet efficiency," a measure intentionally contrived to cause Condorcet's voting system to be best possible, and having, in my opinion, no particular value aside from that. (Indeed, the "Condorcet winner" can be non-best from a utility standpoint, a fact evident in my study's numerically *nonzero* Bayesian regret values in 2-candidate elections. In my studies, this happened $\approx 10\%$ of the time.) Another interesting computer study, but aimed at an entirely different goal (empirical assessment of the probabilities of various kinds of "voting paradox" scenarios, rather than attempting to compare different voting systems) was by Fishburn [17].

These previous computer studies had the following demerits compared to my study here.

1. They all were far smaller than mine, e.g. involving a small subset of my voting systems. My study is the only one with a documented random number generator, and involves ≥ 400 times more simulated elections. (This decreases the statistical margin of error by a factor of ≥ 20. All my regret data have 90%-confidence error bars well below 1%.) Thus it should entirely supercede them.

2. They all got inconclusive results, i.e. were unable to confidently identify any single voting system as "best." On the other hand, *my study concludes that range voting was always best for either honest or strategic, voters in all 144 different election scenarios tried.* This is presumably because none of the previous studies included range voting as a participating voting system. (If I had omitted range voting, I too would have got similar inconclusive results.)

3. Both Merrill's [27] and Bordley's [4] studies only allowed "honest voters," without allowing "strategic

voting." This detracts heavily from any claim those studies can have, to applicability in the real world.

4. Merrill's utility based substudy is suspicious because it was unable to detect the fact that, e.g. 2-candidate majority vote is non-optimal from a utility standpoint, i.e. has nonzero Bayesian regret. (All his data for 2-candidate elections had "100.0% social utility efficiency," in his terminology.) That suggests that Merrill's computer program had bugs.

5. My study involves a superset of the utility-generating methods (see §10.2) used previously, and also is the first (§10.8) to allow "voter ignorance."

It may also be interesting to try to evaluate the merits of voting schemes in genuine human elections, rather than in artificial computer generated ones. However, this is far more difficult and expensive exercise fraught with error and doubt. The largest attempt to do this [15] (based on 37 and 92 contemporary elections within British trade unions and other organizations) concluded that Plurality was worse than 5 other procedures considered but none of these 5 could be clearly favored over any other. These top 5 included Borda, Approval, and Hare-STV. (Range voting was ignored.) It may also be of interest that the Hare-STV system is employed in national elections in the Republic of Ireland, where it is mandated by the (1937) Constitution[29]. In two nationwide referenda (in 1959 and 1968) the Irish were offered the option of abandoning STV, and both times kept it[30].

10.2 The ways I used to assign utilities

The simplest way to assign the c candidate utilities for each voter is simply to use Vc independent random deviates uniform in $[0, 1]$. I call this "random uniform utilities."

Another way, which I call "issue based utilities," is as follows. Each of the c candidates is initially assigned an I-dimensional vector of real numbers in $[-1, 1]$, his "stances on the I issues." Each voter is also assigned such an I-vector. The utility of that candidate for that voter is then the dot product of their two issue-stance vectors, plus a random uniform deviate in $[-1, 1]$. (We may then normalize these utilities by adding c and then multiplying by $1/(2c + 2)$ so that every utility number is in $[0, 1]$; such normalization enables comparison with random utilities.) The "random utilities" method is just the special case of the "issue based" method when $I = 0$.

Issue-based utilities tend, in the limit $I \to \infty$ of an infinite number of issues, to a scenario I call "random normal utilities:" use Vc independent random *normal* deviates (i.e. with probability density $\exp(-x^2/2)/\sqrt{2\pi}$).

In all my experiments I used a combination of the UNIX "random" ("lagged additive Fibonacci" [26] with a 256-byte [i.e. 64 machine word] state array, based on the primitive trinomial $x^{63} + x + 1 \bmod 2$), UNIX "drand48" (iterates $x \leftarrow ax + c \bmod 2^{48}$ where $c = 11$ and $a = 25214903917$), Park-Miller "minimal standard" (iterates $x \leftarrow 48271x \bmod 2^{31} - 1$) and Coveyou nonlinear (iterates $x \leftarrow (x + 1)x \bmod 2^{64}$ where $x \equiv 2 \bmod 4$) generators. I took a 4-way combination by means of bitwise exclusive ORing and modular summation because I did not trust any of these generators individually.

10.3 The voting systems I tried

I programmed 30 different voting systems, listed below, in addition to the "baseline" system, honest true-utility voting, which as we've seen (§4) is the best possible voting system, provided the voters are honest, i.e. it has zero Bayesian regret. (One of the advantages of artificial voters is one can make them honest.)

```
(Baseline=Best-summed-utility winner; regret=0)
0. Honest range voting (scaled utility vote)
1. Honest Borda
2. Honest Condorcet Least-Reversal (CLR)
3. Honest Coombs STV (most least-liked candid
             eliminated each round)
4. Honest Hare Single Transferable Vote STV
       (least most-liked canddt eliminated)
5. Honest Copeland (win most pairwise elections)
6. Honest Dabagh point-and-a-half
7. Honest Black (if no Condorcet winner use Borda)
8. Honest Bucklin
9. Honest plurality+runoff for 2 top finishers
10. Honest plurality (1 vote for max-util canddt)
11. Honest bullet (1 vote against min-util cand)
12. Majority vote on random candidate pair
13. Random "dictator" voter dictates winner
14. Random winner
15. Worst-summed-utility winner
16. Honest approval (threshhold=avg canddt utility)
17. Strategic range/approval (average of
        2 frontrunner utils as thresh)
18. Rational range/approval (threshhold=moving avg)
19. Rational plurality (vote for 1 of 2 frontrnnrs)
20. Strategic Borda I (1 frontrunner top, 1 bottom,
                       rest recursively)
21. Rational bullet (vote against 1 of 2 frontrnnrs)
22. Strategic CLR (strat same as 26)
23. Strategic Hare STV (strat same as 26)
24. Rational Borda (1 frontrunner max, 1 min, rest
    using moving avg to decide if max or min vote)
25. Strategic Coombs STV (strat same as 26)
26. Strategic Borda II (1 frontrunner max,
                   1 min vote, rest honest)
27. Rational Dabagh point-and-a-half (moving avg)
28. Strategic Copeland (strat same as 26)
```

[29] In the 1990 Irish presidential race [8], the first STV round awarded B.Lenihan 44%, M.Robinson 38%, and A.Currie 17% of the first-rank votes. Most Currie supporters prefered Robinson to Lenihan, so that after Currie's elimination, Robinson won the final round with 53%. Presumably in a plain-plurality election Lenihan would have won.

[30] However, this is complicated by the fact the Irish also use a multiwinner form of STV for electing MPs.

`29. Strategic Black (same strat as 26)`

For descriptions of these systems, see §3, §3.1, §9.1, §10.4. For the regret tables output by my computer study, see §10.7.

Systems 0-13 involve "honest" voters, who always order their votes compatibly with their private orderings of the candidate utilities. Indeed, in Gibbard's systems 12-13 (cf. §9.1), the rational voting strategy *is* honesty. Systems 14-15 are intentionally bad voting systems thrown in merely to give the reader some idea of the scale on the regret axis, i.e. system 15 intentionally *maximizes* regret. Note that some of the worst voting systems, such as strategic Bullet (21), are nearly as bad as picking a random winner (14)!

Systems 17-29 involve "strategic" voters who take account of the pre-election polls when deciding how to vote – so their votes may be "dishonest." Since all my randomized utility generators are symmetric under permutations of the candidates[31], pre-election polls would be equally likely to return any ordering of the candidates, so, without loss of generality, I always suppose those polls had concluded that candidate 1 was the frontrunner (most likely to win) candidate 2 was second, candidate 3 third,..., and candidate c last. We also will always assume the poll data describes a *spherically symmetric* Gaussian, simplifying strategy calculation (cf. §7). What I am calling "strategic" voters is not necessarily the same thing as what I have elsewhere called "rational" voters. Rational voters choose the vote maximizing their expected utility in some statistical model of the remaining voters. Strategic voters try to be near-rational, but in the interests of simplicity I have not tried always to find the *exact* optimal vote, sometimes settling for a vote which presumably yields higher expected utility than the honest vote, but not as high as the rational vote. Of course, it is interesting to study the effect of plausible but not optimally rational strategies; indeed that is a large part of the reason to study honest voters. System 16 is actually on the borderline between "honest" and "strategic" voting, since 16 is the most strategic form of honest approval voting, cf. lemma 1, given that no poll-data is known. System 17 is also on the borderline, since each voter always produces an "honest" approval-type (i.e. ± 1-vector) vote, but attempts to do so by choosing his utility threshhold (for deciding whether to vote $+1$ or -1) strategically in view of the poll results.

My strategies 22, 23, 25, 28, and 29 try to be honest when in doubt about the most rational course of action. Therefore, these strategies presumably provide a *lower bound* on the regret of the same voting system with true-rational voters (since, presumably, *more voter honesty results in smaller society-wide regret*). Some "calibration" of how much worse *true* rational voters would make STV, Condorcet-LR, etc., for society, versus my semi-honest voter strategies 22, 23, 25, 28, 29, may be obtained by comparing strategy 26 (and to a lesser extent 20) with true-rational Borda voting method 24.

[31] Thus all my simulated elections were non-generic. This is necessary, of course, to get interesting results, since generic elections are exponentially boring in the limit of large numbers of voters. However, many of my simulated elections had *small* numbers of voters, and thus exhibited large statistical fluctuations causing vast (percentagewise) majorities to be common. Thus our simulations really may be thought of as indicative of what happens in both generic and nongeneric elections.

10.4 More precise description of some of the strategies

Strategic Borda I (system #20): give the best of the two frontrunners the max vote $c-1$ and the worst the min vote 0. Then proceed recursively on the remaining $c-2$ candidates. (This is not the true-rational voter strategy for Borda – for that, use #24).

Strategic Condorcet Least Reversal, Copeland, Black, Hare & Coombs STV (22,28,29,23,25): give the best of the two frontrunners the max vote and the worst the min vote. Order the remaining $c-2$ candidates honestly. This is a plausible-sounding strategy, since it maximizes the chances that the disliked frontrunner will be eliminated in some STV round and minimizes the chances the liked frontrunner will be; similarly this minimizes the number of Condorcet vote-reversals the more liked frontrunner must endure, while maximizing this number for his opposing frontrunner.

System 18: this is the "moving average strategy" for range voting, described in §7; similarly 24 is the moving average strategy for Borda voting, 27 for Dabagh, etc.. Indeed, all the systems I have called "rational" above are just the appropriate versions of the moving average strategy for that COAF system.

10.5 How to obtain my computer program

Internet download from `http://www.neci.nj.nec.com/homepages/wds/votetest.c`.

10.6 The numbers of voters, candidates, and issues

All 144 combinations of parameter possibilities with $V \in \{5, 10, 20, 50, 100, 200\}$, $c \in \{2, 3, 4, 5\}$, and $I \in \{0, 1, 2, 3, 4, \infty\}$ (where $I = 0$ means using random uniform utilities and $I = \infty$ means using Gaussian-normal random utilities) were tried. For each parameter set, a huge number of randomized elections were run to find the 22 expected regrets. Rather than give error bars for each datapoint, I instead have simply tried to run enough experiments to make the errors small. "Bootstrap" tests[32] indicate that 10^5 elections suffice to cause all regret values to have 90%-confidence error bars all better than $\pm 1\%$. But the number of elections I used to obtain each regret datapoint was larger than 10^5; ranging from 666666 with 200 voters, to $2 \cdot 10^7$ with 5 voters. So I believe all my tabulated final regret values have 90%-confidence error bars below $\pm 0.3\%$.

[32] You can also make your own bootstrap error estimates by subtracting twice the RandomWinner regret (system #14) from the WorstWinner regret (#15). Intuitively the RandomWinner regret should have the largest additive error so this should yield an overestimate of every additive error.

10.7 The results

For the full set of regret tables, see the separate data sheets or the electronic version of this paper on my web site[33]. Here are two typical regret tables – for the Monte-Carlo runs with $(V, I) = (20, 0)$ and $(50, 2)$.

Random Utilities. (0 issues.) 20 voters.
Each candidate utility (for each voter)
normalized to lie somewhere in [0,1].
Each regret datapoint averages 4000000 expts.

system	2 canddts	3 canddts	4 canddts	5 canddts
0	0.14203	0.09328	0.06659	0.04941
1	0.14203	0.14661	0.14023	0.13055
2	0.14203	0.18989	0.21179	0.22247
3	0.14203	0.21014	0.25329	0.28478
4	0.14203	0.21853	0.27591	0.32314
5	0.14203	0.17647	0.18645	0.18803
6	0.14203	0.14661	0.19632	0.26643
7	0.14203	0.18771	0.20312	0.20680
8	0.14203	0.26182	0.26740	0.26258
9	0.14203	0.21511	0.32407	0.42706
10	0.14203	0.26106	0.37712	0.48628
11	0.14203	0.29241	0.45289	0.60039
12	0.14200	0.50672	0.74236	0.91523
13	0.56227	0.84417	1.03011	1.16938
14	0.72907	1.09416	1.32965	1.50218
15	1.45800	2.18891	2.66019	3.00443
16	0.14203	0.14041	0.17883	0.20575
17	0.14203	0.22607	0.27907	0.31838
18	0.14203	0.22607	0.27853	0.31554
19	0.14203	0.50697	0.74282	0.91522
20	0.14203	0.50697	0.57875	0.70637
21	0.14203	1.09384	1.32964	1.50184
22	0.14203	0.50691	0.71089	0.86287
23	0.14203	0.50697	0.74282	0.91522
24	0.14203	0.50691	0.64678	0.70219
25	0.14203	0.50691	0.62443	0.57312
26	0.14203	0.79978	1.03564	1.20864
27	0.14203	0.50691	0.74282	0.91522
28	0.14203	0.50691	0.63935	0.75389
29	0.14203	0.50691	0.71089	0.86287

Issue Based Utilities (2 Issues). 50 voters.
Each candidate utility (for each voter)
normalized to lie somewhere in [0,1].
Each regret datapoint averages 2222222 expts.

system	2 canddts	3 canddts	4 canddts	5 canddts
0	0.09374	0.07403	0.06165	0.05368
1	0.09374	0.10418	0.10413	0.10079
2	0.09374	0.12534	0.13967	0.14640
3	0.09374	0.15060	0.18827	0.21523
4	0.09374	0.15432	0.20058	0.23786
5	0.09374	0.12655	0.13838	0.14181
6	0.09374	0.10418	0.15821	0.22782
7	0.09374	0.12308	0.13239	0.13370
8	0.09374	0.26157	0.19968	0.22931
9	0.09374	0.15620	0.24117	0.31057
10	0.09374	0.21097	0.30492	0.37884
11	0.09374	0.27689	0.39935	0.50505
12	0.09381	0.33858	0.49621	0.61039
13	0.41922	0.62885	0.76357	0.86222
14	0.48946	0.73396	0.89227	1.00462
15	0.97846	1.46825	1.78438	2.01238
16	0.09374	0.10419	0.13888	0.16549
17	0.09374	0.15654	0.19923	0.23232
18	0.09374	0.15654	0.19886	0.23101
19	0.09374	0.33825	0.49653	0.61072
20	0.09374	0.33825	0.39428	0.47854
21	0.09374	0.73421	0.89223	1.00606
22	0.09374	0.33834	0.48312	0.58958
23	0.09374	0.33825	0.49653	0.61072
24	0.09374	0.33834	0.44268	0.48438
25	0.09374	0.33834	0.46003	0.41718
26	0.09374	0.53630	0.69482	0.80872
27	0.09374	0.33834	0.49653	0.61072
28	0.09374	0.33834	0.45364	0.54443
29	0.09374	0.33834	0.48312	0.58958

As you can see, *honest* range voting (#0) is the minimum-regret system among the 31 tried, for every column in every table. Among the systems (#16-29) involving *strategic* voters, strategic range voting systems 16-18 also always outperform every other strategic system. All this is true not just in the two tables I've shown here, but also in the 34 other tables (with different numbers of voters and/or different utility generators) I haven't shown.

The regret ratios can be large. For example, the United States, by adopting Plurality voting, is presumably suffering (assuming strategic voters) 2.3-3.0 times as much regret as it could have suffered by using range voting, in 3-5 candidate elections. Assuming honest voters, the USA is suffering 3-10 times as much regret as it could. This is assuming that my 200-voter experiments suffice to get a good enough approximation to the limiting situation with a huge number of voters. If this assumption is wrong, then these regret ratios will (apparently) be even larger and even more in favor of range voting – since the ratios empirically increase with V when $V > 15$, according to the table below:

Random-utility 4-candidate elections. V voters.

	--honest--		regret	--rational--		regret
V	range0	plur10	ratio	range18	plur19	ratio
5	0.03065	0.17480	5.70	0.14688	0.35431	2.41
10	0.04582	0.26249	5.73	0.20410	0.53122	2.60
20	0.06659	0.37712	5.66	0.27853	0.74282	2.67
50	0.10698	0.61238	5.72	0.43077	1.16710	2.71
100	0.15185	0.87716	5.78	0.60457	1.64639	2.72
200	0.21542	1.25863	5.84	0.85423	2.32316	2.72

The regret caused by plurality voting is not just large relative to range voting – it also seems large in any sense:

[33] http://www.neci.nj.nec.com/homepages/wds/works.html

Compare rational plurality system #19 (or even honest plurality, #10) with the election of the worst (#15) or a random (#14) candidate (representing complete failures of democracy). Strategic plurality is only about 3-5 times better than simply electing the worst candidate.

Borda, STV, and Least-Reversal, although much touted, are shown by my studies to suffer sometimes large expected regret ratios versus range systems for strategic (or honest) voters. Keep in mind that our regret values for these methods are *lower bounds*[34] on the true regret since we have studied a voting strategy more honest and less rational than the true-rational strategy. Thus the situation seems even more favorable for range voting than is shown by my regret tables. This effect should be especially pronounced in the case of Coombs (cf. footnote 6).

A peculiarity of the experimental data which immediately worried me is: The two strategic voting systems 17,18 almost always yield the same votes in my experiments (consequently they have regrets identical to $\approx 1\%$), despite the fact that one can construct examples[35] of utility vectors that cause these two systems to yield different vote vectors.

The explanation of this peculiarity is just that such counterexample utility vectors arise rarely. With $c \leq 3$ candidates, strategies 17 and 18 are identical and both are honest. In votes among 4 candidates with random utilities, strategies 17 and 18 yield different vote-vectors only about once every 10 votes, and strategy 18 only "dishonestly" misorders its votes with respect to the true utility ordering, about once every 36 votes[36]. With 5 candidates, strategies 17 and 18 produce different vote vectors more often ($\approx 23\%$ of the time) and strategy 18 produces a "dishonest" vote more often – but still, only about once every 15 votes. (These figures all remain approximately unaltered with issue-based utilities.)

This rarity goes a long way toward explaining why rational range voting (system 18) has so low regret: *strategic range voters are astoundingly honest!*

There is another disturbing numerical pattern: the regrets for methods 19, 22, 23, and 27-29 are always the same to 5 decimal places (at least, if elections with ties are removed). This is because:

Theorem 8 (The winner in 6 voting systems, with strategic voters) *Generically, strategic Condorcet Least-reversal, Black, Dabagh vote-and-a-half, Copeland, and Hare-STV, (strategies 22, 23, 27, 28, 29 in our Monte Carlo study) and rational Plurality voting (system 19) all will yield the same winner in a tie-free V-voter election as $V \to \infty$, namely: the majority winner among the two frontrunners from the pre-election polls, will always be elected.*

Proof: For plurality voting, this was well known. For Condorcet least-reversal: The most popular frontrunner will be top ranked by more than 50% of strategic voters. Hence, he will win all pairwise elections, and hence (since zero reversals will be required) the election. (This is yet another illustration of the fact that making the "Condorcet winner" win, as in property **CW** of §9.3, is not necessarily a good idea.) Consequently the same is true in Black's system and the Copeland system. For Hare-STV: The two poll-frontrunners will garner all the top-rankings from strategic voters in the Hare-STV system 18, thus never being eliminated until the final round, whereupon the most popular one will win. For Dabagh point-and-a-half: The most popular frontrunner will get $> 50\%$ of the votes, which is too great a margin to be outvoted by the combined winners of all the half-votes. QED.

Remark. I consider this theorem very damning testimony against all 5 of these supposed "improved" (versus Plurality) voting systems. Although obvious in hindsight, it seems not to have been noticed previously. It holds, not only for the voting strategy used in this Monte Carlo study, but indeed for *any* strategy which begins by assigning the two poll-frontrunners the maximum and minimum votes (a plausible sounding strategy, which as we've seen in lemma 1 is in fact the true-rational strategy for COAF systems). This theorem also represents yet another nail in the coffin for Bartholdi et al. [1]'s theory that figuring out how to manipulate Hare-STV system by strategic voting would be too difficult, so that voters would not bother and would simply be honest. [Also note: despite my disparaging remarks re the behavior of *Coombs's* STV system in the presence of strategic voters in footnote 6, we now see that in this respect Hare's STV system can be even worse.]

Another important point I should mention (since it significantly affects some statistics): in all my computer simulations, any vote ties were broken *randomly* in such a way that all tied contenders were equally likely to win[37].

Finally: the reader may wonder how it can be that all the voting systems in the Monte Carlo study exhibited nonzero regret even in *two*-candidate elections (where they all, except for silly methods 13-15, reduce to majority vote). The answer is that voter majorities (and the Cordorcet winner **CW**) can be wrong! If 51% of the voters think A is better by 1 utility unit, while 49% think B is better by 97 utility units, then majority vote will elect the wrong candidate: A.

10.8 The effect of voter ignorance – equal and unequal

Steven J. Brams, after reading an earlier version of this paper, worried that "range voting could work well with

[34] Conjecturally. But this conjecture seems very plausible and it is entirely supported by the comparison of system 26 versus the true-rational strategy 24.

[35] Such as $\vec{U} = (0, 9, 100, 11)$ where the candidates are ordered by decreasing likelihood of election according to the pre-election polls. In this case system 12 will vote $(-, +, +, +)$ and system 13 will "dishonestly" vote $(-, +, +, -)$.

[36] Of course, strategy 17 can never misorder its votes, but more rational voters should prefer strategy 18.

[37] In some systems, ties are impossible if the number of voters is *prime*.

well-informed voters. But I'm dubious about less informed voters... half the electorate doesn't know who the vice president is."

Only extremely well-informed voters indeed knew, for example, that President G.W.Bush, soon after his election, would try to rescind rules requiring mine-operators to post bonds to pay for repairing environmental damage caused by that mine – or that President W.J.Clinton would involve himself in sex scandals. Perhaps if voters had known these facts, it would have perturbed their private mental utility values for these candidates in various directions.

To try to address this, I modified my election simulator to allow "ignorant voters." As before, each voter has a true utility for each election winner, and these true utility values are used to assess the post-election Bayesian regrets. *But* now, each voter does *not* know his own candidate-election utilities; instead he knows a version of these values *polluted* by the addition of *ignorance*, i.e., added noise. Specifically we add a Gaussian random deviate with mean 0 and standard deviation Q to every mental utility value before that voter votes[38]. (If $Q = 0$ this reduces to the old, ignorance-free, version of the program.)

As we've just described it, voters have *identical* probability distributions of their ignorance for all candidates. But in practice, some candidates are better understood by most voters than others. To model that, I made a further modification of the simulator in which the Q value, governing the width of the ignorance-perturbation, was now *candidate-dependent*. Specifically Q_j (the value for candidate j) was now itself made a random variable uniform on $[0, Q]$, chosen once per election.

I then re-ran all the $V = 200$ voter elections[39] using both $Q = 0.99$ (a quite-large ignorance value) and $Q = 0.49$, for both candidate-dependent and candidate-independent ignorance. As a typical example of the resulting data, I give the $I = 0$, $Q = 0.99$, Candidate-independent-Ignorance case. (The statistical error for each regret value is $\sigma \lesssim 1\%$.)

In *every* case (both in this $Q = 0.99$, $I = 0$ table, as well as in all the tables with $I \neq 0$ or $Q = 0.49$ or candidate-dependent ignorance not shown here), range voting was still the best voting system, regretwise, for either honest or strategic voters.

```
Random 0-1 Utilities. (0 issues.) Q=0.99. 200 voters.
Each candidate utility (for each voter)
normalized to lie somewhere in [0,1].
Each regret datapoint averages 499999 expts.
```

system	2 canddts	3 canddts	4 canddts	5 canddts
0	1.78246	2.59671	3.12266	3.49234
1	1.78246	2.63211	3.17503	3.55450
2	1.78246	2.65583	3.21910	3.62039
3	1.78246	2.67760	3.26785	3.68912
4	1.78246	2.68080	3.27151	3.69833
5	1.78246	2.66878	3.23158	3.63670
6	1.78246	2.63211	3.23365	3.68527
7	1.78246	2.65211	3.20879	3.59620
8	1.78246	2.74216	3.30105	3.73183
9	1.78246	2.69074	3.33987	3.82141
10	1.78246	2.73779	3.38799	3.86082
11	1.78246	2.74216	3.39982	3.89144
12	1.78002	2.92934	3.68618	4.23271
13	2.25672	3.39143	4.11966	4.65564
14	2.31529	3.45491	4.21275	4.74960
15	4.60974	6.91339	8.40655	9.49843
16	1.78246	2.63539	3.21484	3.63070
17	1.78246	2.69523	3.28534	3.72672
18	1.78246	2.69523	3.28480	3.72029
19	1.78246	2.92934	3.68535	4.22196
20	1.78246	2.92934	3.55226	4.05111
21	1.78246	3.46370	4.19826	4.75376
22	1.78246	2.92810	3.67800	4.20765
23	1.78246	2.92934	3.68535	4.22196
24	1.78246	2.92810	3.61878	4.06401
25	1.78246	2.92810	3.68921	3.97279
26	1.78246	3.19770	3.94175	4.48897
27	1.78246	2.92810	3.68535	4.22196
28	1.78246	2.92810	3.66088	4.17900
29	1.78246	2.92810	3.67800	4.20765

[38] Allowing non-zero means would have made no difference if these means were all candidate-independent. If they themselves were candidate-dependent (but identically distributed independently sampled) random variables, then that would merely have had the effect of making the statistical noise worse, without affecting the relative rankings of the voting systems by expected regret.

[39] With $(I, c) \in \{0, 1, 2, 3, 4, \infty\} \times \{2, 3, 4, 5\}$.

Although the fact of range voting's superiority seems to remain unchanged in the presence of ignorance, many other things are altered by $Q = 0.99$ ignorance:

1. Essentially every voting system now exhibits much larger regret – up to 10 times larger.
2. The differences among voting systems (expressed as regrets) are *relatively* much smaller. For example in the table above, all regrets from non-silly systems were within a factor of 1.3 of each other, whereas without ignorance, the spread was a factor of 10.

These two trends also both are present to a lesser extent in the $Q = 0.49$ data.

10.9 Still more (lesser known) voting systems

After I made this study available on the internet, it attracted considerable attention from the popular press, political science professors, and voting reform advocates. Some of the latter suggested other voting systems and/or various reasons why Range Voting might not be as good as I thought. I was then able to respond to these criticisms by adding extra features to my voting simulator and re-running the experiments. I will summarize this here. (In §10.8 I have already mentioned Brams's "voter ignorance" worry and what my study said about it.)

Voting reform advocate Blake Cretney[40] likes N.Tideman's [38] "ranked pairs" variant of the Con-

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dorcet least-reversal system better than the variant I studied. In the Tideman system, you pick the $A > B$ comparison with the largest margin and "lock it in." Then you pick the next largest one available ("available" means: not already used and not creating a cycle in the directed graph of candidate-comparisons) and continue on. This ultimately creates an ordering of the candidates. The topmost in the ordering wins. The Tideman system is equivalent to my own Condorcet variant if there are $c \leq 3$ candidates; but if $c \geq 4$ they can differ. Tideman's system is much faster than mine (polynomial time as opposed to NP-complete) if used in multiwinner elections, but it is slower in 1-winner elections. Cretney was kind enough to email me computer code implementing this algorithm. Systematic experiments then showed that my own Condorcet variant is superior – although only by about 5% – to Tideman's (as far as Bayesian Regret is concerned) when $c = 4$ and $c = 5$ with honest voters. With strategic voters as in my method #22, both are identical.

In reaction to this, Cretney remarked that
(a) Tideman's method was not designed with the intent of minimizing Regret, but instead to feature simultaneously "Monotonicity" and "Independence of Clones." See [38].
(b) The Condorcet-LR variant that I prefer (which was argued by Young [37] to be the one Condorcet himself wanted; see my footnote 7) exhibits this perhaps-disturbing property: if it is used to produce a full rank-ordering of all candidates, then the top-ranked candidate may not be the same as the winning candidate if it had only been used with the intent of producing 1 winner.

Independence of Clones (**IC**) means: if you have a set C of "cloned" candidates, and remove one candidate X from all ballots, then if some member of C was the winner, some member of C should still be the winner after X's removal; similarly if some candidate not in C won before, he should still win after. The idea is to give a precise notion of avoiding "vote-splitting" and the opposite phenomenon (which Cretney calls "teaming") where a party gets rewarded for running many identical candidates (e.g. the Borda system tends to reward this). Cretney notes that "most ranked methods fail **IC**." Note, however, that range voting and Hare-STV both pass **IC**. (Proof sketch for Hare-STV: each time a clone is eliminated his votes move to one of his co-clones, so that ultimately the battle is between just 1 representative from each clone-class.)

James Gilmour[41] suggested I look at Brian L. Meek's version [23] of the Hare-STV system – advocated by the British Electoral Reform Society and used in their elections as well as those of the Royal Statistical Society and London Mathematical Society. However, after considerable effort in programming and testing Meek's system, I realized that it is *identical* to Hare-STV for elections with only 1 winner (it can differ if there are ≥ 2 winners), i.e., the only kind studied in this paper!

Terrill Bouricius[42] favors the Hare-STV (he calls it IRV) system. In a letter to *Science* magazine [8] criticizing a pro-Approval-Voting editorial by Brams and Herschbach, he and 2 others gave the following interesting argument for the presumed superiority of IRV over Range and Approval voting[43]

> "Imagine an AV [Approval] election with 100 voters... after 98 ballots are counted, the results are $Z = 55$, $Y = 60$, $X = 61$. The 2 remaining ballots are cast by voters who love Z and hate X. If they knew these [98-vote totals] in advance they'd block X by voting for Y and Z. But if the [98-vote totals] instead were $Z = 60$, $Y = 61$, $X = 55$ then they'd want to elect Z by *not* voting for Y."

The perceived problem here is that the final two Approval voters would be unhappy because, *in the absence of information* about the other 98 votes, they would not know which voting strategy to use[44]. In short, the question is: what will happen if voters want to vote strategically but suffer from a lack of information about what the other voters are doing? Will this result in more unhappiness (on average) with Approval or Range[45] voting, versus IRV?

So Bouricius believed that IRV was better. He had a 4-part argument (which I do not agree with) for why:

1. In practice, in the absence of polls, voters will often be ignorant about what the other voters are doing.
2. Because of that ignorance, voters may make wrong strategic voting decisions (which they will regret bitterly), or may be unable to think of any good voting strategy at all.
3. Therefore they will vote honestly.
4. Therefore IRV will work well.

My initial reaction to this argument was that it has several flaws. Bouricius thought (2) was only an attack

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[43] In this letter, they also – falsely – claimed that an IRV "voter's best strategy is to sincerely rank the candidates." I gave a counterexample in §9.4 in the discussion of Hare-STV's violation of **H3**.

Brams and Herschbach counterattacked, incidentally, by noting that IRV often punishes centrist candidates in 3-candidate elections by eliminating them in round 1 so that one of the two extreme candidates then wins. It would seem more reasonable in such scenarios for the centrist to win – which Range and Approval voting would do.

[44] Bouricius, with some justification, does not agree with my over-simplistic notion that, thanks to pre-election and during-election polls, voters usually *do* have such information – he observes that for most, e.g. municipal, elections in the USA, such polling data usually is *not* available. But one might counter-argue that it is usually a good bet that the two frontrunners will be the Republican and Democrat, thanks to some kind of psychological carryover from more-major elections.

[45] Incidentally, Bouricius's scenario seems to be a good argument for the superiority of *Range* voting over Approval – the Range voter is not forced into one of these two choices but can instead award Y and intermediate number of votes.

on approval voting, but it seems to me it *also* attacks IRV voting and in fact attacks essentially any voting system. I do not agree that (3) necessarily follows from (2) because even working in a lot of ignorance one often can still think of a dishonest voting strategy that sounds likely to be better than honesty – in *both* IRV and Range voting. Finally, my computer simulations show that honest voters do not make IRV work as well as they make range work (as evaluated by Bayesian Regret scores), so I think (4) is *not* a pro-IRV argument, it is a pro-Range argument.

Nevertheless, Bouricius could be more right than wrong, since (2) and (3) do not *have* yes/no answers, they only have numerical answers (i.e.: how often this happens and how much it matters) which are not obvious a priori. It could be that these numbers happen to have the net effect of favoring IRV over Range voting in the presence of poorly-informed strategic voters.

Fortunately, a careful look back at my **computer simulation data provides a clear refutation** of this whole pro-IRV argument.

We simply compare Bayesian Regret values of the following 3 voting systems:

```
A. Honest IRV (=Hare-STV) voters (system #4)

B. Honest plurality voters
   (#10; I put these in as a "baseline.")

C. Strategy-16 approval/range voters
```

Note re (A): Bouricius believes that IRV voters who are ignorant about what the other voters are doing, will be unable to think of a good strategy and hence will vote honestly. (If Bouricius is wrong – i.e. voters will think of dishonest voting strategies anyway – my regret data will presumably merely yield a *lower bound* on regret since dishonest voters should cause more regret than honest ones. Therefore the results of my data will be artificially *biased in Bouricius's favor* by some unknown amount.)

Note re (C): This strategy is designed to work well in conditions of ignorance about what the other voters are doing since it tends to maximize the expected impact of your range vote. ("Impact" = probability it will break a tie, times the utility difference between the 2 tied candidates.) See lemma 2.

The computer simulation data is clear: C is *always* superior to A which in turn is *always* superior to B, in all simulated elections *except* the ones with only 5 voters. (In 5-voter elections A sometimes is superior to C by very small margins, e.g. with utilities based on 2 issues, A was better than C with 4 candidates by 0.00009 and with 5 candidates by 0.00052. These regret-difference advantages are small enough that they may be due to statistical noise.)

Considering that this data is artificially biased in Bouricius's favor and *still* invalidates his argument, I think we can safely dismiss that argument despite its intuitive appeal.

Voting reform advocate G.A.Craig Carey[46] devised a voting system he calls "IFPP3" for conducting *three*-candidate elections. Let the 3 candidates be A, B, C. You can cast exactly one of 9 possible types of votes:

```
The 9 kinds   Name of variable to
of votes     count that kind of vote
-----------  -----------------------
  A>{B,C}              a0
   A>B                 ab
   A>C                 ac
  B>{A,C}              b0
   B>C                 bc
   B>A                 ba
  C>{A,B}              c0
   C>A                 ca
   C>B                 cb
Let
  a = a0+ab+ac;   b = b0+ba+bc;   c = c0+ca+cb;
```

If all voters express *full* preference orderings (i.e. only 6 of the 9 kinds of votes are now allowed so a0=b0=c0=0) then A wins an IFPP3 election if and only if

$$\begin{gathered}(b+c<a)\wedge\\ [(b+cb<a+ca)\vee(2b<a+c)]\wedge\\ [(c+bc<a+ba)\vee(2c<a+b)]\,.\end{gathered} \tag{11}$$

(The conditions for B or C's victory are symmetric. Carey does not actually specify precisely what to do about ties, but if the number of voters is prime, ties are impossible.)

Result: after adding IFPP3 to the computer simulator (for both honest and strategic voters), the results were: For 1-winner 3-candidate elections IFPP3 is better than plain plurality, but not by a lot; it is definitely inferior to Hare-STV; and the best system remains Range Voting.

Dwork, Kumar, et al. [14] invented several interesting new voting systems incorporating algorithmic ideas much more sophisticated than are in the usual voting systems. (Namely: one of their systems finds the minimum-weight perfect matching in a bipartite graph; another constructs a Markov chain and finds its stationary limit ditribution by solving an eigenproblem; and they also have an idea they call "local Kemenyization" which can be used to "improve" the output-ordering gotten from almost any other voting system.) They then did experiments concluding that their system "MC4" was superior to all other systems they tried, and in particular superior to Borda by "a huge margin."

My reaction to Dwork, Kumar, et al. [14] is this. First, all their experiments can simply be dismissed as not statistically significant. They nowhere in their analysis used error bars or any statistical confidence bars, and all of their experimental conclusions are based on a total of only 38 elections – by comparing the voting systems's output orderings of the candidates with their

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subjective human judgement of what the "best" such ordering "should" have been. Even their "huge margin" of superiority of MC4 versus Borda – about a factor of 2 – still may be due to chance, since the event that at least twice as many heads as tails occur in any 38 coin flips, happens by chance about 2% of the time. Second, their MC4 scheme is very similar to the Copeland system – it depends on pairwise comparisons only and ignores the "margins of victory" in those comparisons – and thus almost certainly (by *my* measures of voting system performance) should be of rather low quality.

Third, despite the above, I think their ideas are promising and plausibly *could* yield voting systems comparable to or slightly better than Condorcet's method. As a typical example of their Markov ideas, consider the following Markov chain M: move from candidate A to candidate B if a random voter prefers B to A, otherwise stay at A. Find the stationary limit-distribution D_M of M (by finding the unique eigenvector with positive eigenvalue of its $c \times c$ transition matrix). Then they order the candidates according to their decreasing probabilities in D_M, the "winner" being the one with the greatest limit-probability.

10.10 Which is the best?

It seems clear that range voting is experimentally best. For either honest voters (systems #1-15) or strategic ones (systems #14-29), range voting produces smaller Bayesian regret, in *all* 144 election scenarios (with different numbers of voters, candidates, and utility generating methods) tried, and this is also true with either of 2 levels, in 2 models, of artificially-induced voter "ignorance."

There seem to be three possible escape hatches to try to invalidate this conclusion:

1. If a large fraction of voters had somehow become falsely convinced that honesty is the best voting strategy in the Condorcet-LR system – but they correctly understood Range voting strategy – then since honest-CLR (#4) exhibits smaller (up to 30% smaller) regret values than rational-Range (#18), CLR would become the best system. This is conceivable, since I myself once suffered from that delusion about CLR (and also Hare-STV)!

2. Since the best voting strategies for CLR and Hare-STV are not known, it is conceiveable that, if in future these true-rational strategies become known, then redoing my computer study using voters using these strategies, would entail lower regrets than Rational range voting. I tend to doubt that because the strategies (#23, 22) I *did* test for these systems ("honesty except about the two frontrunners") seem likely to be more honest and hence less regret-causing than whatever the true rational strategies are. (I think the best chance for overthrowing Range in this way would be in 3-candidate elections; see open question #2 in §11.)

3. With future utility generating methods, perhaps the relative regrets of the various voting systems might change. The dominance of Range voting is so great under my present utility generators, that its overthrow seems unlikely, but it is possible. Since I have made my computer program available on the web (§10.5) it will be easy for others to plug in their own utility generators to investigate this possibility.

Let me now discuss these in more detail.

Hare-STV with honest voters (#4) also outperforms (although only by 4-6%) strategic range voting #17 and #18 in 3-candidate elections with 0 or ∞ issues, although both these range voting systems are superior for 1,2,3 and 4 issues. (With more candidates, Hare-STV no longer looks as competitive – with 5 candidates it always is inferior to all 3 range voting strategies #16,17,18. Also, strategy #16 for Range voting is always the best with $V \geq 10$ voters even when $c = 3$.) So *overall*, the *regret* data prefers strategic range voting over honest Hare-STV. Because STV voting also has severe *algorithmic* disadvantages versus all additive voting systems[47] and versus Condorcet (and because honesty is not the best voter strategy in Hare-STV 3-candidate elections anyway), I think Hare-STV may safely be removed from contention for the title of "best voting system."

So the only contenders for the Title are Condorcet-LR and Range – and CLR can only compete if the voters are deluded.

But even if the voters *are* deluded about voting strategy, for 3-candidate elections the question of whether Range voting would be superior to Condorcet in practice would hinge upon what *percentage* of range voters would vote honestly or would use strategy #16 instead of #18 – since honest Range voting, and strategy-16 range voting, both are superior regret-wise even to honest-CLR. (In practice I would expect voters would use some mix of strategies, including both optimal and non-optimal ones, and including honesty.) Range voting also has these advantages over CLR: it is simpler to explain, analyse, and run, and it is superior (by larger margins than 30%!) in the presence of enough honest (or strategy-16) voters, and there is no looming uncertainty about what happens in elections with $c \geq 4$ candidates (where the best voter strategy for Condorcet is unknown).

Both Range and Condorcet-LR are unquestionably better than plurality (as good or better regret values in all 144 election scenarios tried).

10.11 Why is Range Voting superior to Approval Voting?

Here is an argument made to me by fans of approval voting: Generically, rational range voters will vote in an approval manner, i.e. will always choose vote values at the endpoints (here ± 1) of the allowable range. So, if one

[47] See discussion of **Wk** in §9.3; a close STV election among a large number of candidates would be a nightmare for election administrators dwarfing the Florida 2000 nightmare!

believes that that only rational voters exist, then the two systems generically will be equivalent. If so, why incur the extra algorithmic complexity of range voting?

Here is my counterargument:

1. I expect that *some* fraction of honest voters will exist. (I estimate 10%.) They should be given the opportunity to be honest! As an example of how that can pay, consider the following scenario by D.Saari: 999 voters prefer $A > B \gg C$, while 1 voter says $C > B \gg A$. If everyone approves their top two candidates, then B is elected, extremely contrarily to the will of the majority. If however we adopt range voting with at least 1% of the voters being honest and voting $(1, 0.9, 0)$-style, and the remaining ones strategic and voting approval-style, then A is elected and life is good.

2. Suppose that there are a large number of candidates that most voters know nothing whatever about[48] Then most voters in an approval election would probably "play it safe" by disapproving of the unknowns. However, it would be better if they could assign them all an "average" value (corresponding to their perception of the quality of the average person), perhaps decreased by some "safety margin." This is possible in range but not in approval voting. Without this possibility, tremendous biases against lesser-known canidates will be introduced artificially by the voting system.

Consequent practical recommendation: I recommend allowing range voters to specify their numerical votes for some subset of the candidates and then to say "... and give each of the remaining candidates X" (where X is some value they specify).

11 Conclusion and Open problems

The main *experimental* contribution of this paper has been the first utility-based large Monte Carlo comparison of different voting systems – with the conclusion that range voting utterly dominates all other systems tried, both for honest and for strategic voters. Roughly: range voting entails 3-10 times less regret than plurality voting for honest, and 2.3-3.0 for strategic, voters. Meanwhile strategic plurality voting in turn entails only 1.5-2.5 times less regret than simply picking a winner randomly.

The main technical *theoretical* contribution of this paper has been the notion of the "generic election" in the limit $V \to \infty$ of a large number V of voters, and the recognition that it drastically simplifies voter-strategy analysis, permitting characterization both of optimal expected-utility voter strategies, and of "uniquely best" voting systems.

The issues discussed in this paper are totally independent of the puzzling question of how to allow accurate and verifiable, yet cryptographically secure and (optionally) anonymous, voting, in which vote-buying is impossible. For the application of "zero knowledge proof techniques" to resolve these apparently irreconciliable desiderata, see the beautiful paper [29].

Neither will Range (nor Condorcet-LR) voting solve such societal problems as unwise voters[49] and innaccurate vote tallying[50].

[48] This was surely the case in the 2003 California Governor recall election won by A.Schwarzennegger, since there were over 100 candidates.

[49] For example, R.M.Nixon was re-elected US president in 1972 by a "landslide:" he got 61% of the vote, the second highest percentage ever recorded. Essentially every voting system would have declared him the winner. However, since Nixon then became the only president to resign, with most of the key members of his cabinet jailed on corruption charges, most people now would agree this was not the best possible decision.

[50] At present in the USA (judging by the results of numerous contradictory vote recounts and re-recounts during the Gore-Bush contest, cf. footnote 4) it seems to be impossible to run elections with expected fractional error below 10^{-3} or 10^{-4}. Many would say that, really, even 10^{-3} is highly optimistic. For example a 29 Dec 2000 *New York Times* article by David Stout indicated that Gore's nationwide *popular* vote lead over Bush had climbed to 539947 (versus the figure of 337000 widely reported in the month after the election). This margin shift was ≈ 0.002 of the total number $V \approx 105 \times 10^6$ of votes cast. The intense examination of the Bush-Gore *Florida* contest cast some light on the difficulties involved: **(1)** The number of ballots rejected as invalid in the Florida 2000 election (about 2.5% of the total votes cast; 3% is typical in contemporary USA elections) exceeded Florida's Bush-Gore margin of 537 votes by a factor of ≈ 300. The legal criteria for rejecting ballots are vague. **(2)** This 537-vote margin also appeared to be about 5 times smaller than the number of *accidental* mis-votes for Buchanan in Palm Beach County, Florida [9]. **(3)** An examination by *The Miami Herald* [22 Jan. 2001] uncovered over 2000 illegally-cast ballots in Florida including felons, 3-year dead corpses, and double voting (and there probably were far more) – exceeding the official Bush-Gore margin by a factor of 4. It is impossible to tell for whom these ballots were cast. This is important because it indicates that determining the winner was actually mathematically impossible. **(4)** Statistical studies conducted by *Miami Herald*, *USA Today*, *Washington Post*, and *Knight Ridder News Service* (and, according to press reports, by the Gore and Bush teams themselves), all concluding a statewide recount probably would have led to a Gore victory. This is mainly because the most pro-Gore counties had voting machines of types which have relatively high rates of undercounting of ballots. (For example, when 4695 Palm Beach [a Pro-Gore county] ballots were hand tallied, 33 new votes were found for Gore and 14 for Bush. When 3892 ballots were hand-counted in Broward [a pro-Bush county], 6 new votes were found.) The *Miami Herald*'s estimate based on optical scans of ballots was that Gore would have won by 23,000 votes. **(5)** A media consortium [10] including the *New York Times*, *Wall Street Journal*, *Washington Post*, *CNN*, and *Associated Press*, spent >$1 million to count the 175010 as-yet uncounted (since, e.g. rejected by ballot machines) ballots in the Florida election by hand using various criteria for accepting ballots. The results, announced over 1 year later (NY Times, 12 Nov. 2001): If Gore's request to recount 4 counties had not been blocked by the US Supreme Court, Bush still would have won Florida by 225 votes. Furthermore, if all Florida ballots had been recounted under vote-acceptance standards specified by each of the 67 counties, Bush still would have won, by 389 to 493 votes (depending on whether a 3-0 or 2-1 vote of 3 ballot examiners was demanded for deciding what each ballot meant). However, if instead of using 67 different standards, one of 8 proposed *uniform* standards had been used ("hanging chads," "penetration," etc.), then Gore would have won by 15 to 424 votes. **(6)** The *Post* on 27 Jan 2001 reported that an examination of computer files of 2.7 million invalidated "overvotes" from Miami-Dade,

Open question #1: For what kinds of voting systems are pre-election polls capable of being reliable? ("Bullet" voting, as we've seen in §8, seems to be a counterexample, if there are a large number of rational voters – there I called voting systems in which rational voters will act to invalidate any pre-election poll claims, "suicidal.")

Open question #2: Determine the best voter strategy (in my model) in the Condorcet Least Reversal system for an arbitrary number c of candidates. I suspect this should be possible using a similar analysis to mine, but involving Gaussian distributions and decision surfaces in a $(c-1)c/2$-dimensional space rather than a c-dimensional space. Use this understanding to redo the computer simulation comparing CLR to strategic range voting.

Open question #3: How can this generic/rational picture be extended to try to find uniquely best systems for electing w winners chosen from among c candidates for *general* w, $1 \le w < c$ (possibly with $w \ne 1$), where each voter has $\binom{c}{w}$ utility values in mind, one for each of the possible election results? What about if those $\binom{c}{w}$ utilities just arise as *sums* of only w out of c mental utility values? I think the ideas in the present paper will be a key part of the answer to this question, but considerably more work will be needed.

12 Acknowledgements

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Broward, Palm Beach, Hillsborough, Pinellas, Marion, Highlands and Pasco counties, showed that 46000 had Gore as one of the choices, while 17000 had Bush. So if these counties had employed "instant check" voting machines allowing voters to instantly detect and correct illegal overvoted ballots, presumably Gore would have won by about 28000. **(7)** A later *Miami Herald* study of 64248 "undervotes" from all 67 Florida counties concluded (10 March 2001) hand-counting them would have increased Bush's lead over Gore to 1655. The same study, though, found that hundreds of ballots were thrown out in Palm Beach and Volusia counties that had marks no different from ballots deemed valid – Gore would have been elected if they had been counted. **(8)** [Orlando Tribune 7 May 2001, article by Damzon & Roy] Florida county officials on election day made over 10000 copies of mismarked or torn absentee ballots, which probably helped Bush (who later came out against all "human interpretation of voter intent" on ballots). **(9)** Democratic party lawyers collected over 20,000 affadavits from Florida voters claiming that their votes had in various ways been denied or impeded. (E.g. black people carpooling to polling places stopped by police asking for a "taxi license.") This exceeds the Bush-Gore margin by a factor of 37. The U.S.Civil Rights Commission weighed in on 9 March 2001 by reporting the Florida election had been marred by "significant and distressing" barriers that discouraged qualified voters from casting ballots; e.g. they found that many blacks did not vote because they were assigned to polling sites that lacked adequate resources to confirm their eligibility. The Media Consortium's statistics showed vastly larger ballot rejection rates in predominantly black precincts.

[51] `http://www.barnsdle.demon.co.uk/vote/sing.html`

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[6] S.J.Brams & P.C.Fishburn: Approval Voting, American Political Science Review 72,3 (September 1978) 831-847; also book with same title, Birkhäuser Boston 1983.

[7] Steven J. Brams and Peter C. Fishburn: Some logical defects of the single transferable vote, Chapter 14, pp. 147-151, in Choosing an Electoral System: Issues and Alternatives (Arend Lijphart and Bernard Grofman, eds.) Praeger, New York 1984.

[8] S.J.Brams & D.R.Herschbach: The science of elections, [Editorial] Science 292,5521 (25 May 2001) 1449. Pro-IRV letter in reply from R.Richie, T.Bouricius, P.Macklin: Science 294,5541 (12 Oct 2001) 303-304; response: 305-306.

[9] Statistical studies placed on the web by various professors re the Buchanan vote anomaly in Palm Beach, Florida:
Greg D. Adams:
`http://madison.hss.cmu.edu/palm-beach.pdf`;
Henry Brady:
`http://socrates.berkeley.edu/~ucdtpums`;
Christopher D. Carroll:
`www.econ.jhu.edu/people/ccarroll/carroll.html`;
Craig Fox:
`http://faculty.fuqua.duke.edu/~cfox/Bio/election2000note.pdf`;
Bruce Hansen:
`http://www.ssc.wisc.edu/~bhansen/vote/vote.html`;
M.C. Herron, W.R. Mebane, Jr., J.N.Wand, K.W.Shotts, J.S. Sekhon:
`http://elections.fas.harvard.edu`;
John S. Irons:
`http://www.amherst.edu/~jsirons/election/`;
B.L.Monroe:

http://www.indiana.edu/~playpol/pbmodel.pdf; Till Rosenband: http://web.mit.edu/norstadt/Public/election.pdf; Michael Ruben: http://weber.ucsd.edu/~mruben/florida.htm. All of these reached essentially the same conclusion: about 2800 Gore votes accidentally went to Buchanan.

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