Cruel and unusual behavior of the Riemann zeta function

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Abstract — **We exhibit a sequence c_n such that the convergence of $\sum_{n\geq 1} c_n z^n$ for $|z|<1$ is equivalent to the Riemann Hypothesis. Numerical investigation of the c_n revealed some astonishingly deceptive behavior.**

Keywords — Riemann zeta function, analyticity.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \tag{1}$$

if $\Re(s) > 1$, and by analytic continuation for other complex s. $\zeta(s)$ has a simple pole at $s = 1$, but is analytic everywhere else in the complex s plane. The "critical line" is $\Re(s) = 1/2$. The "Riemann Hypothesis," which is one of the most important open problems in mathematics, is the conjecture that all of the zeroes of $\zeta(s)$ in the region $\Re(s) \geq 1/2$ lie on the critical line. A tremendous amount of information about $\zeta(s)$ and about various statements implied by, implying, or equivalent to the Riemann Hypothesis, or about the known evidence and partial results on it, may be found in the references.

A very simple attack on the Riemann Hypothesis occurred to me. Observe that the conformal map $z = 1 - 1/s$, $s = 1/(1-z)$ makes the region $\Re(s) > 1/2$ correspond to the unit disc $|z| < 1$. So consider the function

$$F(z) = \ln\left[\frac{z}{1-z}\zeta(\frac{1}{1-z})\right]. \tag{2}$$

This function has been intentionally designed so that the Riemann Hypothesis is equivalent to the statement that $F(z)$ is analytic in the unit disc $|z| < 1$. And this in turn is equivalent to the statement that the Maclaurin series

$$F(z) = \sum_{n\geq 1} c_n z^n \tag{3}$$

converges when $|z| < 1$. (It is easy to see that the c_n are real and that $c_0 = 0$.)

Lo and behold, c_1, ... c_{25} are positive and strictly decreasing! Obviously, if these two facts were to continue forever, that would immediately imply the Riemann Hypothesis. And furthermore, notice that if you pick 25 random real numbers (e.g. from any probability density symmetric about zero) then the probability that by luck they are going to be positive and decreasing is $2^{-25}/25! \approx 1.9 \times 10^{-33}$. This is smaller than the probability that of picking a given air molecule from all the ones in the room. So, as any physicist could tell you (?), it must be the case.

But, in fact, it is not the case, as I discovered when I wrote a program to compute the first 150 c_n. The first increase is $c_{28} \approx 0.022801390 < c_{29} \approx 0.022937613$. Although this is a small increase, it is genuine.

But, examining these 150 coefficients, I noticed that although not all of them are decreasing, still, all of them are positive. And the conjecture that all the c_n are positive alone is enough to imply the Riemann Hypothesis. The proof is simple. Assume the RH is false, so some point of nonanalyticity of $F(z)$ exists within the unit circle $|z| < 1$. In that case, since the $c_n > 0$, it would be the case that there must be a singularity of $F(z)$ actually on the *real* interval $(0, 1)$. But, no such singularity exists (obvious from the definition of $\zeta(s)$ for real $s > 1$). QED. Now furthermore, if you pick 150 random numbers, the probability that they, by luck, are all going to be positive, is $2^{-150} \approx 7.0 \times 10^{-46}$. OK, maybe 1.9×10^{-33} was not small enough, but *now*, surely, we have enough confidence. This conjecture must be true, right? Visions of sugarplums danced in my head as I imagined ways to try to prove the c_n are all positive. For example, one can write down various closed forms for the c_n, and then try to prove them positive.

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But in fact, this conjecture is also false. I found this out by computing the first 2048 c_n; the first negative one is $c_{156} \approx -0.000139116$. The c_n then monotonically get more negative until reaching a min at $c_{172} \approx -0.005993512$, and then monotonically increase until reacvhing a max at $c_{217} \approx +0.017704939$ and then monotonically decrease until reaching the most negative coefficient that I know of (indeed, it may be *the* unique most negative coefficient... a conjecture which also would imply the Riemann Hypothesis![1]), $c_{266} \approx -.008839076$.

Numerical values, rounded to 5 decimal places, for $c_1,\ldots, c_{299}$ are in table 1. These were computed by evaluating $F(z)$ at 2048 points uniformly spaced around a circle of radius 128/129, and then estimating c_n by means of the Cauchy residue theorem with the integrals being evaluated numerically by means of the trapezoidal rule, i.e. a "fast fourier transform." I cannot claim to have a proof that every decimal in the table is correct, but I consider it highly likely because the results were checked by similar computations with different numbers of points and different radii of the circle and while carrying different numbers of decimal places (in the present computation I carried 60 decimal places); and also the first 25 coefficients were computed by an entirely different algorithm (transformation of a known series for $\zeta(s)$) and used as a check. c_1 is the Euler-Mascheroni constant $\gamma = .57721566490153286\ldots$, a fact which also may be used as a check. I also used two different languages, `MAPLE` and `MATHEMATICA`. The latter seems to have a far superior (both in speed and accuracy) arbitrary precision implementation of $\zeta(s)$.

One may plot c_n versus n for $0 < n < 2048$, and the resulting picture looks like a continuous curve which oscillates in a random-looking manner with a rough period of about 100 and with an apparently gradually decreasing amplitude.

By transforming well known facts about $\zeta(s)$ one may show that

$$F(z) = \frac{K}{1-z} - \ln 2 - \ln\Gamma(\frac{3-2z}{2(1-z)}) + \sum_{\kappa} \ln\frac{\kappa - z}{1-z} \tag{4}$$

where $K = \ln(2\pi) - 1 - \gamma/2 + \sum_\kappa (1-\kappa)$ and κ are the locations of the non-real singularities of $F(z)$. Also, $F(z)$ obeys the functional equation

$$F(z) = F(\frac{1}{z}) + \ln(\frac{-z}{\pi}) + \frac{1}{1-z}\ln(2\pi) + \ln\sin\frac{\pi}{2(1-z)} + \ln\Gamma(\frac{-z}{1-z}). \tag{5}$$

To conclude:
(1) These quantities (the c_n) deserve more study, since any at most subexponentially increasing bound on them would imply the Riemann Hypothesis.
(2) It's remarkable how the Riemann zeta function seems to be trying *intentionally* to deceive us!

1 Acknowledgement

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2 References

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[1] Because if all coefficients obeyed $c_n > -c$, then $F(z) + c/(1-z)$ would have only positive Maclaurin series coefficients, and then a similar proof to the one above, would imply analyticity for $|z| < 1$.

$a \backslash b$	0	1	2	3	4	5	6	7	8	9
0		.57722	.48344	.40690	.34390	.29165	.24805	.21146	.18061	.15451
1	.13238	.11359	.09762	.08406	.07256	.06284	.05465	.04779	.04208	.03737
2	.03353	.03045	.02802	.02615	.02477	.02380	.02318	.02287	.02280	.02294
3	.02324	.02367	.02419	.02479	.02543	.02609	.02675	.02739	.02801	.02858
4	.02910	.02956	.02994	.03024	.03047	.03060	.03065	.03061	.03048	.03026
5	.02996	.02957	.02909	.02854	.02792	.02722	.02646	.02564	.02477	.02385
6	.02289	.02189	.02087	.01982	.01876	.01770	.01662	.01556	.01450	.01346
7	.01244	.01144	.01048	.00956	.00868	.00784	.00706	.00632	.00565	.00503
8	.00447	.00398	.00355	.00318	.00289	.00266	.00249	.00240	.00236	.00240
9	.00249	.00264	.00286	.00312	.00345	.00382	.00423	.00469	.00519	.00572
10	.00629	.00688	.00749	.00812	.00877	.00942	.01008	.01073	.01139	.01203
11	.01266	.01327	.01387	.01443	.01497	.01547	.01594	.01637	.01676	.01710
12	.01740	.01765	.01785	.01799	.01808	.01812	.01811	.01803	.01791	.01772
13	.01749	.01720	.01686	.01647	.01602	.01554	.01500	.01443	.01381	.01316
14	.01247	.01176	.01101	.01025	.00946	.00866	.00784	.00702	.00619	.00536
15	.00453	.00371	.00290	.00211	.00134	.00059	−.00014	−.00084	−.00150	−.00213
16	−.00271	−.00326	−.00376	−.00422	−.00463	−.00498	−.00529	−.00554	−.00574	−.00588
17	−.00596	−.00599	−.00597	−.00588	−.00574	−.00555	−.00530	−.00500	−.00464	−.00424
18	−.00379	−.00329	−.00275	−.00217	−.00156	−.00090	−.00022	.00049	.00123	.00199
19	.00277	.00356	.00436	.00517	.00598	.00679	.00760	.00840	.00918	.00995
20	.01070	.01143	.01213	.01280	.01344	.01405	.01461	.01514	.01562	.01605
21	.01644	.01678	.01707	.01730	.01749	.01761	.01769	.01770	.01767	.01757
22	.01743	.01722	.01697	.01666	.01630	.01589	.01544	.01494	.01439	.01381
23	.01318	.01252	.01183	.01110	.01035	.00957	.00878	.00796	.00714	.00630
24	.00545	.00461	.00376	.00291	.00208	.00125	.00043	−.00036	−.00114	−.00189
25	−.00262	−.00331	−.00398	−.00461	−.00520	−.00576	−.00627	−.00674	−.00717	−.00755
26	−.00788	−.00816	−.00840	−.00858	−.00872	−.00880	−.00884	−.00882	−.00876	−.00865
27	−.00849	−.00828	−.00803	−.00773	−.00739	−.00701	−.00659	−.00613	−.00565	−.00512
28	−.00457	−.00400	−.00340	−.00277	−.00213	−.00148	−.00081	−.00013	.00056	.00125
29	.00194	.00263	.00331	.00399	.00465	.00530	.00594	.00656	.00716	.00773

Table 1: Values of c_{10a+b} to 5 decimal places.