The drag on a sphere moving through a blackbody radiation bath

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Abstract — **A black or mirror sphere of surface area A moving at velocity v in a bath of thermal radiation at temperature T will experience a drag force $A\sigma T^4 v/(c^2 - v^2)$ where c is the speed of light and σ is the Stefan-Boltzmann constant.**

This is a corrected version of a previous TN.

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1 Drag on a large massive sphere

Suppose a sphere of surface area A is moving at velocity v thru a vacuum containing a blackbody radiation bath at temperature T. Assume the sphere radius is much larger than the thermal wavelength, the sphere is sufficiently massive that it is not set into vibration by the incident waves, and that we are considering time scales much larger than the gaps between collisions with single photons.

Claim: *The sphere will experience a drag force (written here in the reference frame of the sphere)*

$$\kappa A\sigma T^4 v/(c^2 - v^2) \tag{1}$$

where $\kappa = 1$ for either a black or a mirror sphere.

Note: We can replace the "geometrical optics" assumptions with other ones, by altering the drag formula appropriately, see §2.

Derivation: The idea is that the sphere sees a blue shifted (to the front, or red shifted, to the rear) blackbody bath, and hence experiences a net force. One way to derive this is to start with the Doppler shift formula

$$\omega_{\text{dest}} = \omega_{\text{source}} \frac{\sqrt{1 - v^2/c^2}}{1 - v/c \cos\alpha} \tag{2}$$

where α is the angle (in the destination frame) between direction of light and velocity of the light source, and perform lots of integrals and Lorentz transformations. But there is a simpler way.

From the point of view of the sphere, the whole blackbody bath is Lorentz contracted. The blackbody bath has an energy density. In the sphere frame, since the radiation is contracted, it has more energy density. Also, that energy is moving, so more energy still. The product of these two Lorentz factors is $c^2/(c^2 - v^2)$.

Energy still leaks into the sphere at the same fractional (i.e. relative to the bath energy that would be in the sphere) rate, though.

If the energy density in the bath is mc^2, then the momentum density is mv, that is, v/c^2 times the energy density. The rate of energy leakage would be $A\sigma T^4$ where A is the sphere's surface area, if it were stationary, so the rate of momentum leakage is $A\sigma T^4 v/(c^2 - v^2)$. And that is the blackbody drag for a black sphere. The correction factors for other kinds of spheres are derived in the next section.

Since the radiation is not isotropic in angle, due to Lorentz contraction, the reader may be worried about our claim that the fractional energy leakage rate is independent of v. Although we have different energy densities at different angles, by integrating over angle we are getting the true energy density, and also getting the true energy leak rate. This trick works specifically because we have a *sphere*, which permits the whole argument to be deceptively simple. Therefore the formula is correct.

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2 Cross sections

What about if we have a different kind of scatterer (for instance, a small, or non-black, sphere) so that our assumptions above are not valid? In that case, one will have to correct the drag formula above by multiplying by the "momentum thermal cross section" of the scatterer and dividing by the momentum thermal cross section πa^2 of a black sphere of "large" radius a.

In this section, we compute various cross section formulas for that purpose.

The differential cross section, averaged over polarizations, for a small but massive perfectly conducting sphere of radius a, in a beam of light of wavelength λ, $a \ll \lambda$, is (Jackson 1975, §9.6c)

$$\frac{d\sigma}{d\Omega} = \frac{a^6}{\lambda^4}[\frac{5}{8}(1+\cos^2\theta) - \cos\theta] \tag{3}$$

and if this is integrated *times* $1-\cos\theta$ (for the momentum contribution) $2\pi\sin\theta d\theta$ for $\theta \in [0,\pi]$, you get

$$\sigma_{\rm mom} = \frac{14\pi a^6}{3\lambda^4}. \tag{4}$$

This is not the same as the Mie-Debye 1909 formula (16.159 in Jackson) for the "total cross section" σ, which is not appropriate for average momentum transfer. (The drag on a particle in a light beam hereby is $\sigma_{\rm mom}$ times the momentum flux in the beam.)

If this is integrated times the Planck energy density (in λ) of 3D blackbody radiation at temperature T, i.e. times

$$\frac{2\pi hc^2}{\lambda^5(e^{hc/(\lambda k_B T)}-1)}d\lambda \tag{5}$$

from 0 to ∞, and the result is divided by the total integrated energy density $2\pi^5 k_B^4 T^4/(15h^3c^2)$, we get for the "thermal momentum cross section"

$$\sigma_{\rm mom}^T = \frac{112\pi^5}{3}\frac{a^6 k_B^4 T^4}{h^4 c^4}. \tag{6}$$

Because a moving sphere actually sees different *Doppler shifted* Planck distributions at different angles, the use of this formula to correct (EQ 1) will only be valid in the nonrelativistic limit.

The differential cross section, averaged over polarizations, for a small but massive transparent sphere with index of refraction η, is (Jackson §9.6b, eqn 9.88)

$$\frac{d\sigma}{d\Omega} = \frac{a^6}{\lambda^4}\frac{1}{2}|\frac{\eta-1}{\eta+2}|^2(1+\cos^2\theta). \tag{7}$$

It then follows that

$$\sigma_{\rm mom} = \frac{a^6}{\lambda^4}\frac{8\pi}{3}|\frac{\eta-1}{\eta+2}|^2, \tag{8}$$

$$\sigma_{\rm mom}^T = \frac{64\pi^5}{3}|\frac{\eta-1}{\eta+2}|^2\frac{a^6 k_B^4 T^4}{h^4 c^4}, \tag{9}$$

which are both the same formulas as for the perfectly conducting small sphere, but multiplied by $\frac{4}{7}|\frac{\eta-1}{\eta+2}|^2$.

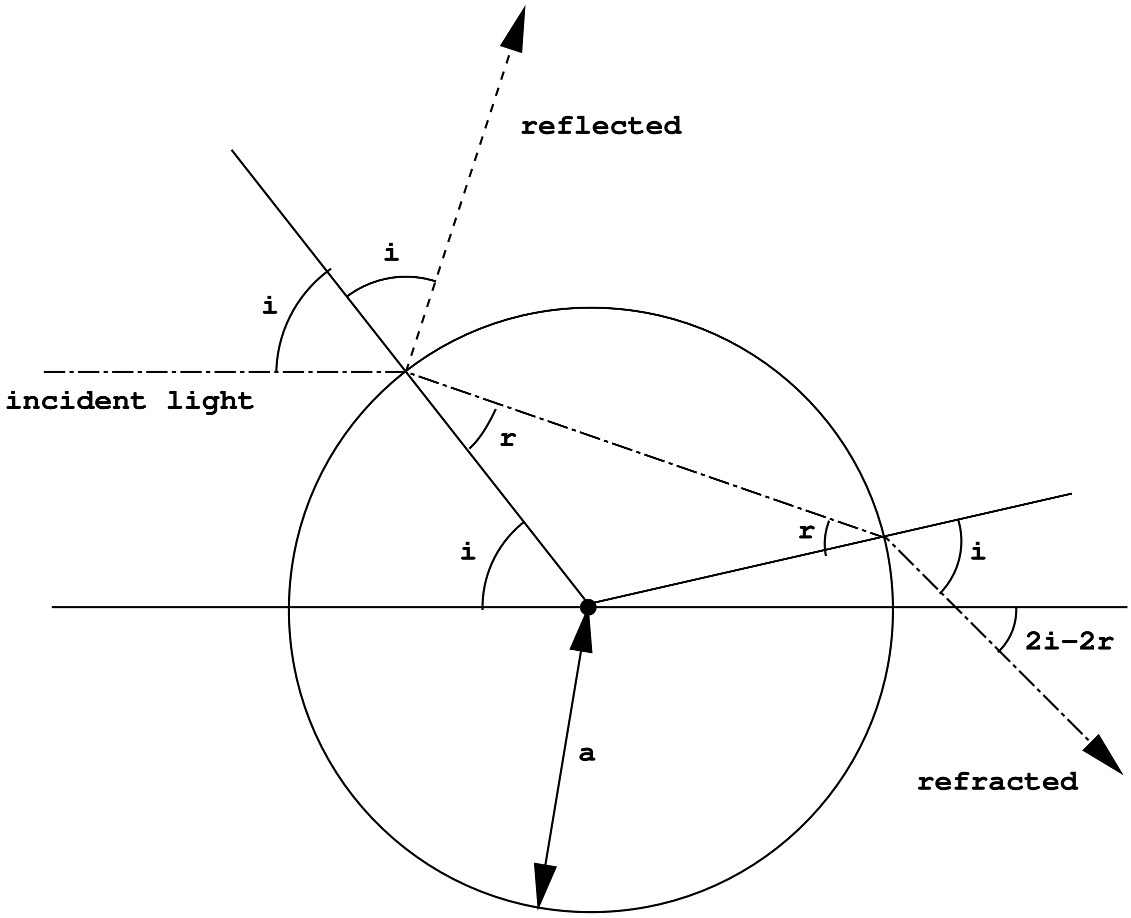
For a large ($a \gg \lambda$) massive mirrored sphere of radius a,

$$\frac{d\sigma}{d\Omega} = a^2/4 \tag{10}$$

by geometric optics. See the figure. Specifically, the cross sectional element is $a^2\frac{\sin i}{\sin\theta}\cos i di d\phi$, where the $\cos i$ comes from the tilt of the area element by an angle i from normality, and the $\sin i/\sin\theta$ is a correction factor in the multiplier $\sin\theta$ of $d\phi$ when doing the angular integration. Using the geometric fact $\pi - 2i = \theta$, this is $\frac{\cos\theta\sin\theta/2}{2\sin\theta}a^2 d\theta d\phi$ which simplifies to the expression above. The momentum integral yields

$$\sigma_{\rm mom} = \pi a^2. \tag{11}$$

This happens to be the same as σ (and not twice it, as would have happened in the case of a mirror *disk*), and due to λ-independence $\sigma_{\rm mom}^T = \sigma_{\rm mom}$. This leads to the intuitively suprising result that the drag on a mirror sphere is the same as the drag on a black sphere.



Light incident on a sphere of radius a at a point at angle i from the centerline. If the light is refracted, the net effect will be to bend the light's direction by angle 2i−2r, where r and i are related by Snell's law. If the light is reflected, the net effect will be to bend the light's direction by 180 degrees minus 2i.

Figure 1:

For a large ($a \gg \lambda$) transparent sphere of radius a, consult the figure. By geometrical optics using Snell's law of refraction $\sin i = \eta \sin r$, we see that the total bend angle of light incident on the sphere at angle i, is $\theta = 2i - 2r$, and hence the total momentum cross section is given by

$$\sigma_{\text{mom}} = 2\pi a^2 \int_0^{\pi/2} \frac{\sin i \cos i}{\sin \theta} [1 - \cos \theta] \, \mathrm{d}i. \tag{12}$$

where $\theta = 2i - 2r$ and $\sin i = \eta \sin r$. I do not know how to evaluate this integral in closed form, and indeed do not know how to write the differential cross section except implicitly by writing $i(\theta)$ in terms of the solution of a transcendental equation. However, in the limit when $\eta \to \infty$, so that $\theta = 2i$, and also in the limit $\eta = 1 + \epsilon$, for small $\epsilon > 0$, when $\theta = 2\epsilon \tan i + O(\epsilon^2)$, we may evalute the integral. The approximate formula

$$\sigma_{\text{mom}} \approx |\frac{\eta - 1}{2\eta}| \pi^2 a^2 \tag{13}$$

happens to be exact in both these cases. Note that for large η, this cross section approaches $\pi^2 a^2/2$, which is actually greater than πa^2, the momentum cross section of a black or mirror sphere, but of course not greater than $2\pi a^2$, the maximum possible value, achieved by a mirror disk.

For Thomson scattering of light by an electron (valid in the nonrelativistic limit)

$$\frac{d\sigma}{d\Omega} = a^2 \frac{1}{2} (1 + \cos^2 \theta), \tag{14}$$

where a is the "classical electron radius" $r_e = e^2/(m_e c^2)$. (The same formula would hold, of course, for a nonelectron of charge q and mass m if we use $a = q^2/(mc^2)$ in place of r_e.) The integral yields

$$\sigma_{\mathrm{mom}} = \frac{8\pi a^2}{3} \tag{15}$$

which happens to be identical to σ, and also $\sigma^T_{\mathrm{mom}} = \sigma_{\mathrm{mom}}$. Observe that this is the same formula as (EQ 11) except for a factor of 8/3.

The Thomson formula is the nonrelativistic limit of the Klein-Nishina formula (see Feynman 1961) which states (in a form averaged over all polarizations) that the differential cross section for scattering of incident light of wavelength λ by an initially stationary electron, resulting in scattered light of wavelength γ going in a direction bent by angle θ, is

$$\frac{d\sigma}{d\Omega} = a^2 \frac{1}{2} (\frac{\lambda}{\gamma})^2 (\frac{\lambda}{\gamma} + \frac{\gamma}{\lambda} - \sin^2\theta). \tag{16}$$

If we integrate times $1 - \frac{\lambda}{\gamma}\cos\theta$ (for the momentum contribution) $2\pi \sin\theta d\theta$ for $\theta \in [0, \pi]$ and using the fact (from conservation of energy and momentum) that $\cos\theta = 1 - \gamma/a + \lambda/a$, we get

$$\sigma_{\mathrm{mom}} = \frac{\pi\lambda(\lambda + a)}{3(\lambda + 2a)^3 a^2} [18a\lambda^4 + 102a^2\lambda^3 + 186a^3\lambda^2 + 102a^4\lambda - 20a^5 \tag{17}$$
$$+ (9\lambda^5 + 60a\lambda^4 + 141a^2\lambda^3 + 126a^3\lambda^2 + 12a^4\lambda - 24a^5) \ln \frac{\lambda}{\lambda + 2a}].$$

This is indeed equivalent to (EQ 15) when $\lambda \gg a$. It is also possible to find σ^T_{mom} in closed form, but considering its use to correct (EQ 1) would only be valid in the nonrelativistic limit where we already know σ^T_{mom}, this would be pointless.

3 Consequences

An amusing consequence of our main claim is, that if a sphere has rest mass m_0 and starts life with a velocity v_0, then the total distance ℓ that it will travel, even after an infinite time, is bounded by

$$\ell < \frac{v_0 m_0 c^2}{\kappa A \sigma T^4}. \tag{18}$$

Equality is approached in the nonrelativistic limit $v_0 \ll c$. For example, if we launch a black ball of diameter .1m and mass 1Kg into interstellar space, no matter at how great a speed $v_0 < c$, and supposing that it is traveling through a blackbody bath with $T = 300$ Kelvin and feels no forces besides light drag, the total distance that it will ever traverse would be $\ell < 4.6 \times 10^{23}$m, i.e. about 50 million light years. If in place of this ball, we instead used an electron, see (EQ 15), the maximum achievable distance would be only 2×10^{20}m, i.e. it can't even get 1% of the (2MLyr) distance to the Andromeda galaxy. This effect (still using $T = 300$K but now with $v_0 = 1000$m/s) shows that the Earth's moon will complete at most 5×10^{16} orbits. However, a more realistic temperature in all these estimates is probably 3 Kelvin, in which case they would all be multiplied by 10^8.

The magnitude of drag due to inelastic collisions with stationary matter filling the (imperfect) vacuum at a density ρ would be

$$\pi a^2 \rho v^2 c^2/(c^2 - v^2), \tag{19}$$

the drag force again as viewed in the reference frame of the sphere (of radius a) but where ρ is the density in the background reference frame. (Note, if the background matter were randomly *moving* at velocities $\gg v$, then this drag would only be proportional to the first power of v.)

The density of the universe is supposed to be at most 5×10^{-28}Kg/meter^3, approximately the equivalent of 1 atom of hydrogen per three cubic meters. Using $\sigma \approx 5.67 \times 10^{-8}$Wm^{-2}K^{-4}, $c = 299792458$m/s, and with $T \approx 2.7$K as the temperature of the universe, we see that light drag is larger than collision drag precisely when $v < 4\sigma T^4/(\rho c^2) \approx 10^{-3}c \approx 270000$m/s. This crossover velocity is invariant as the universe expands, assuming that the mass density ρ and the thermal energy density $4\sigma T^4/c$ decrease proportionately. We conclude that this effect (light drag) is sometimes relevant. It may influence the formation of stars and galaxies.

Another way in which this effect may make itself felt is in the ultra-relativistic limit, in which the drag law may be written

$$\dot{E} \approx -\kappa A \sigma T^4 \frac{E^2}{E_0^2} \tag{20}$$

where $E \gg E_0 = m_0 c^2$ is the total energy of the particle. If we use $\kappa A = 32\pi r_p^2/3$ where $r_p = 1.53 \times 10^{-18}$m is the classical proton radius, according to Thomson scattering, and $T = 2.7$K, we see that ultrarelativistic protons would lose their energy on a time scale (in the lab frame) of $\frac{3 m_p c^2}{32\pi r_p^2 \sigma T^4} \approx 6.35 \times 10^{29}sec\approx 2.01 \times 10^{22}$Yr divided by their energy (expressed as a multiple of their rest energy). For this reason, proton energies larger than 10^{16} times rest mass would be cut off over intergalactic distance scales of 2 MLYr.

Our use of the nonrelativistic Thomson scattering cross section here is still valid *classically* even for protons with 10^{12} times their rest energy, since the energies of the 2.7K photons, even Doppler shifted by 10^{12}, are still well below the proton rest energy. However, quantum mechanically, protons with energies over 10^9 times their rest mass will see some thermal photons over the e^+e^- pair production threshhold, so that this classical cross section is an underestimate and large cosmic rays should be cut off more rapidly than this. Indeed, according to calculations by K.Greisen 1966, pion production should cut off protons with energies over 10^{11} times their rest mass over distances of 32 MLyrs. Photon-induced nuclear disintegration should cut off nucleonic cosmic rays with energies over 10^{10} times the proton rest mass on distance scales of only 20 KLyrs.

Unfortunately, the paper by Hayashida et al. 1994 was brought to my attention. Several cosmic rays with energies over 10^{11} times the rest mass of a proton have now been observed. The record (Bird et al. 1994) is a cosmic ray, "consistent with a proton," with an energy of 3×10^{11} proton masses, i.e., over 50 Joules! These rays seem to have come from outside of our galaxy. Everybody concerned agrees that they are very mysterious.

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