The uniqueness of the maximum and minimum functions for AND and OR in fuzzy logic

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Abstract — **Many functions of two variables have been proposed to extend the AND, OR, and NOT functions of logic from $\{0,1\}$ to $[0,1]$. We give uniqueness theorems showing that the min, max, and $(1-x)$ functions are the only choices which satisfy certain conditions. What a logician would regard as very weak conditions will force AND$=\min$ and OR$=\max$, but all properties of interest to logicians are actually satisfied by a very large class of NOT functions.**

Keywords — Fuzzy logic, uniqueness, functional equations.

1 Introduction

Fuzzy logic is the extension of two valued $\{0,1\}$ logic to the continuum $[0,1]$ by generalizing the AND, OR, and NOT functions so that they map $[0,1]^2$ (or $[0,1]$, in the case of NOT) to $[0,1]$. In principle, nearly any function $f(x,y)$ with $f(0,0)=0$, $f(0,1)=f(1,0)=f(1,1)=1$ is a candidate for a fuzzy OR. However, requiring f to obey certain properties greatly reduces the number of possibilities. We will show that fairly weak requirements, motivatived by trying to preserve as many manipulative properties of logic as possible, force the choice $\mathrm{OR}(x,y)=\max(x,y)$, $\mathrm{AND}(x,y)=\min(x,y)$. The choice for $\mathrm{NOT}(x)$ is much more flexible. Still, the usual simple choice $\mathrm{NOT}(x)=1-x$ may be forced by requiring $\mathrm{NOT}(x-\frac{1}{2})-\frac{1}{2}$ to have odd symmetry,

These choices have been used before [7], indeed they are one of the most common selections, but we have not seen uniqueness theorems of the sort we present.

2 A list of possible properties

What sort of properties do we wish for fuzzified AND, OR, and NOT to obey? Here is a list of possible properties; each of the uniqueness theorems we will prove later will assume some (small) subset of these.

1. Idempotency (of AND and OR): $f(x,x)=x$.
2. Associativity ("): $f(x,f(y,z))=f(f(x,y),z)$.
3. Commutativity ("): $f(x,y)=f(y,x)$.
4. Boolean agreement: when the arguments of AND, OR, or NOT assume values in $\{0,1\}$, then their values agree with the usual ones.
5. Monotonicity in each argument of AND and OR with the other held fixed: $f(a,y)\geq f(b,y)$ if $a\geq b$.
6. A strict monotonicity condition for OR: $f(x,y)$ is a strictly increasing function of x when $x\geq y$ and y is fixed, and a strictly increasing function of y when $y\geq x$ and x is fixed. (Similar condition for AND: change $\geq$ to $\leq$ in preceding sentence.)
7. Monotonicity of the NOT function: $h(a)\geq h(b)$ if $b\geq a$.
8. Continuity: $h(x)$ and $f(x,y)$ are continuous functions.
9. Weakened Lipshitz condition for AND and OR: $|f(x,y+b)-f(x,y)|\leq|b|$, $|f(x+a,y)-f(x,y)|\leq|a|$.
10. Lipshitz condition for NOT: $|h(x+a)-h(x)|\leq|a|$.
11. Concavity: AND is a concave-$\cap$ function of x and y, OR is a concave-$\cup$ function of x and y.

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12. Lack of inflection points (a weakened version of the previous condition): As their argument(s) move along any line, the sign of the concavity of AND, OR, and NOT does not change.

13. Odd symmetry of $\mathrm{NOT}(x-\frac{1}{2})-\frac{1}{2}$: $h(1-x)+h(x)=1$.

The following formulae will be written in the language of set theory, but their generalizations to logic and fuzzy logic should be clear. ($\cap$ is AND, $\cup$ is OR, prime is complementation.)

14. Involution: $A''=A$. That is, $h(h(x))=x$, where h is the NOT function.

15. De Morgan's laws: $A'\cap B'=(A\cup B)'$, $(A'\cup B')=(A\cap B)'$.

16. Distributive laws: $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$, $(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$.

17. Equivalence: $(A'\cap B')\cup(A\cap B)=(A'\cup B)\cap(A\cup B')$.

So far, the choice AND= min, OR= max, and $\mathrm{NOT}(x)=1-x$ obeys all the conditions above, and as we will see later it is the only choice which does. A condition obeyed in Boolean logic and set theory, but *disobeyed* in fuzzy logics, is

18. Excluded middle and noncontradiction: $A\cap A'=\emptyset$, $A\cup A'=\mathbf{1}$.

If the fuzzy logic values were to be interpreted as *probabilities* whose correlations are unknown, then one would be justified in demanding that the following condition holds:

19. Range: $\min(x,y)\le\mathrm{AND}(x,y)\le\max(0,x+y-1)$, $\quad\max(x,y)\le\mathrm{OR}(x,y)\le x+y$,

However, many proponents of fuzzy logic have renounced claims to be making, or trying to make, probabilistic statements. Fuzzy logic cannot hope to give true probabilistic values because too much information (about correlations) is unavoidably thrown away by restricting oneself to two-argument fuzzy logic functions. For example, the choice $\mathrm{AND}(x,y)=xy$, $\mathrm{OR}(x,y)=x+y-xy$, $\mathrm{NOT}(x)=1-x$, motivated by regarding the arguments as probabilities of independent events, finds itself predicting $\mathrm{AND}(x,x)=x^2$, contrary to property 1, and also disobeys the equivalence identity 17.

Still, should one wish to have any connection whatever to probability, one would probably want to keep assumption 19.

3 Philosophy

Philosophically, theorems based on subsets of these assumptions fall naturally into two classes, which are respectively of more interest to the "logician" and the "statistician."

The logician wishes for fuzzy logic to be maximally manipulable, and to support a large number of theorems analagous to theorems in ordinary logic, thus justifying using the word "logic" in the name "fuzzy logic." Thus he wishes to preserve as many of the properties 1-4 and 14-18 of ordinary logic as he can. The logician would also support the monotonicity property 5, would be slightly less happy with the continuity property 8, which he would regard as coming from real analysis and not from logic, would look upon a concavity requirement with suspicion (although he might be attracted to it on the grounds of simplicity and elegance) and would be totally skeptical of a need for the lipschitz condition 9.

On the other hand, if one plans to use fuzzy logic, not for the purpose of proving theorems, but instead for the purpose of heuristic reasoning in the presence of uncertainty, then our justification should instead be at a lower level. The fact that fuzzy logic is idempotent and associative according to properties 1 and 2, is of little interest to the statistician, since it could well be that by sacrificing *exact* associativity and idempotence, better performance in some other regime might be obtained. She would also criticize the imposition of 1 and 2 by noting that (for example) $f(x,x)=x^2$, or even x^3-x^2+x, is an equally valid fuzzification of the notion of "idempotence," but leads to radically different results. Of course, from the standpoint of the logician, such alternative statements would be disagreeable since they are not as "simple" for him to use. The statistician, however, doesn't care how "simple" the logician thinks his life is, and cares only about which statement gives better correspondence to the real world. The statistician is handicapped, however, by the fact that she doesn't actually know what the real world is, nor is she sure what constitutes "better performance." She does, however, quite confidently demand that properties 3, 5, 7, 8, and 14 should hold, since she has commonsense arguments for them. For example: changing one's confidence in x by a microscopic amount should not have a macroscopic effect on one's confidence in $\mathrm{OR}(x,y)$, hence the demand for continuity in property 8. After some thought, the statistician would (we claim) also be willing to accept property 6 as an axiom.

The statistician has less approval of the symmetry property 13 and the metrical properties 9 and 10, although she can still offer fairly convincing arguments in their favor. Thus 13 is justified if one wishes that truth and falsity be symmetrical notions. A justification of 9 is: altering one's confidence in x by a small amount a ought not to alter one's confidence in $\mathrm{OR}(x,y)$ by an amount larger than a.

Finally, the statistician views the concavity properties 11 and 12 with suspicion; she cannot see what property 11 "really means" somehow. The weaker property 12 makes some sense – if you are walking along a line in fuzz-space, it seems acceptable for some logical function to bend up or bend down, but implausible that it should do both – but nevertheless makes one uncomfortable.

The uniqueness theorems in this paper should make the logician entirely happy since only a small number of assumptions, all of which he entirely approves of, are needed to force the uniqueness of fuzzy logic.

However, we have not been able to devise theorems which will make the statistician entirely happy, although some of them come close. This is because even by assuming all the properties the statistician wholy approves of, plus throwing in a few more, does not suffice to make fuzzy logic unique. Specifically, properties 3,4,8, strengthened monotonicity (strictly monotonic on all lines with increasing x and y coordinates), and a full strength 2-dimensional lipshitz condition all put together still leaves open the choice $\mathrm{OR}(x,y) = a\min(x+y,1) + (1-a)\max(x,y)$ where a, $0<a<1$, is an arbitrary constant.

4 Uniqueness of the NOT function

Lemma 1 *The general monotonic solution of the functional equation*

$$h(h(x)) = x \tag{1}$$

in the interval $[0,1]$ *is as follows. Let* c, $0<c<1$, *be an arbitrary constant, Let* $g(x)$, $0 \le x < c$, *be an arbitrary continuous strictly decreasing function with* $g(0)=1$ *and* $g(c)=c$. *Then either* $h(x)=x$ *or*

$$h(x) = \begin{pmatrix} g(x) & \text{if} & 0 \le x \le c \\ g^{-1}(x) & \text{if} & c \le x \le 1. \end{pmatrix} \tag{2}$$

For proof, see [4] and [6, Satz 3].

Corollary 2 *The only increasing solution of* $q(q(x)) = x$ *is* $q(x) = x$.

Equation (EQ 1) is a special case of "Babbage's functional equation" $h^{[n]}(x) = x$ when $n=2$. The equation was first studied by computer pioneer Charles Babbage [1, 2] and has since been studied by many authors.

Anyway, we now plainly see

Observation 3 *To force* $h(x) = 1-x$ *for* $0 \le x \le 1$, *it is not sufficient merely to require that* h *obey* $h(0)=1$, $h(1)=0$ *(property 4) and be continuous (property 8), involutory (property 14) and strictly monotone decreasing (property 7).*

Indeed, note that if $g(x)$ is concave-∪ and decreasing, then so is $g^{-1}(x)$, so that any function h of the form (EQ 2) automatically obeys property 12, too. Finally, if AND= min and OR= max, then properties 15, 16 and 17 hold with any function of the form (EQ 2). Hence these properties also are no re¡striction. It is therefore essential to use an extra assumption such as the odd symmetry condition 13 or the Lipshitz condition 10. Either will suffice:

Theorem 4 *If* h *obeys the assumptions of observation 3 and also the odd symmetry condition (property 13) then* $h(x) = 1-x$.

Proof. We have by oddness $1 - g(x) = g^{-1}(1-x)$. Let $q(x) = 1 - g(x)$ to rewrite this as $q(x) = q^{-1}(x)$, so that $q(q(x)) = x$ and q is an increasing function... hence by the "corollary" above, $q(x) = x$. □

Theorem 5 *If* $h(0)=1$, $h(1)=0$, *and* h *obeys the lipshitz condition (property 10) then* $h(x) = 1-x$.

The proof of theorem 5 is trivial. Theorem 4 was stated without proof in [3, p. 13].

We remark that by use of lemma 1, it may easily be shown that the involutory *rational* functions which obey $h(0)=1$ and $h(1)=0$ are precisely Sugano's [5] fuzzy NOT functions $h(x) = (1-x)/(1-kx)$ where k, $k<1$, is any fixed real.

5 Uniqueness of the OR and AND functions

Theorem 6 *Associativity (property 2), idempotency (property 1), and the strict monotonicity condition (6) for OR suffice to force* $f(x,y) = \max(x,y)$. *Similarly using the strict monotonicity condition (6) for AND suffices to force* $f(x,y) = \min(x,y)$.

Proof. By associativity and idempotence

$$f(x, f(x, y)) = f(f(x, x), y) = f(x, y). \tag{3}$$

Now a strict monotonic function assumes any value at most once, so that, if $g(y)$ is strictly monotone in some region R, then $g(a) = g(b)$ and $a, b \in R$ implies $a = b$. Here, regarding x as fixed and y as varying, the region R is $y \geq x$ and a is y and b is $f(x, y)$, which indeed lies in R since $y \geq x$, idempotency, and monotonicity,...so that

$$f(x, y) = y \quad \text{when} \quad y \geq x.$$

Similarly $f(x, y) = x$ when $x \geq y$ so that $f(x, y) = \max(x, y)$. The proof is exactly similar for AND= min. □

The weakness of the assumptions needed is quite remarkable; we have not needed to assume boolean agreement, commutativity, or continuity. However, weakening our assumptions even a tiny bit further is enough to destroy the validity of the theorem. Consider the function

$$F_1(x, y) = \epsilon \max(x, y) + (1 - \epsilon) J(x, y)$$

where

$$J(x, y) = \left(\begin{array}{ccc} \max(x, y) & \text{for} & x, y < \frac{1}{2} \\ \min(x, y) & \text{for} & x, y \geq \frac{1}{2} \\ \frac{1}{2} & \text{otherwise} & \end{array} \right.$$

was devised by Joe Kilian. This function F_1 is continuous, strictly monotone, and idempotent, and, while it is not exactly associative, it is associative to within arbitarily small numerical error ϵ. Despite this, this function radically differs from $\max(x, y)$. This extreme sensitivity is also quite remarkable. We also mention that Vincze [6, Satz 1] has shown, in a result which almost, but not quite, contradicts ours, that the most general continuous associative functions $f(x, y)$ which are *strictly* monotone in each argument, are

$$f(x, y) = \phi^{-1}(\phi(x) + \phi(y)) \tag{4}$$

where ϕ is continuous and strictly monotone, but otherwise arbitrary. Of course, $\max(x, y)$ cannot be written in this form. Thus for example $F_2(x, y) = \sqrt[n]{x^n + y^n}$, $n > 0$, is associative, continuous, and monotone, but not idempotent. Indeed no function of the form (EQ 4) can either be idempotent or exhibit boolean agreement with the OR function. In the limit $n \to \infty$, $F_2 \to \max$.

If the strict monotonicity condition is removed, and replaced with the weaker monotonicity condition of property 5, then theorem 6 becomes false ($J(x, y)$ is a continuous counterexample) and even if we demand boolean agreement, it still remains false – a discontinuous counterexample OR-function is

$$F_3(x, y) = \left(\begin{array}{ccc} 1/3 & \text{if} & x \leq 1/3 \leq y < 2/3 \\ 1/3 & \text{if} & y \leq 1/3 \leq x < 2/3 \\ \max(x, y) & \text{otherwise.} & \end{array} \right.$$

However, demanding both boolean agreement and continuity makes it become true again:

Theorem 7 *Properties 1,2,4,5,8 force OR*$(x, y) = \max(x, y)$, *and similarly AND*$(x, y) = \min(x, y)$.

Proof. By monotonicity $f(1, y) = 1 = f(x, 1)$. Consider $q(y) = f(x, y)$ for fixed x when $x \leq y \leq 1$. By (EQ 3) we have $q(q(y)) = q(x)$. If $q(y) = y$ for $x \leq y \leq 1$ we are done. Suppose $y_1 = q(y_2) < y_2$. Since $q(y_1) = q(y_2)$, by monotonicity $q(y) = y_1$ when $y_1 \leq y \leq y_2$. For $y_3 > y_2$, $q(y_3)$ cannot be less than $q(y_2)$ by monotonicity, nor can $q(y_2) < q(y_3) < y_1$ since that would imply $q(y) = q(y_3)$ when $q(y_3) \leq y \leq y_3$, and in particular $q(y_2) = q(y_3)$, a self-contradiction. Hence by continuity $q(y) = q(y_2) < 1$ when $y_2 \leq y \leq 1$. But $q(1) = 1$; so the initial supposition that $q(y_2) < y_2$ is not possible. By similar reasoning, $q(y_2) > y_2$ is not possible. Hence, $f(x, y) = y$ when $x \leq y \leq 1$, and similarly $f(x, y) = x$ when $y \leq x \leq 1$, so that $f(x, y) = \max(x, y)$. The argument for AND instead of OR is similar. □

Theorem 8 *Idempotency (property 1), boolean agreement (property 4), monotonicity (property 5), and weakened lipshitz (property 9) suffice to force OR*$(x, y) = \max(x, y)$ *and AND*$(x, y) = \max(x, y)$.

Proof. The behavior on the lines $x = y$, $x = 0$, $x = 1$, $y = 0$, $y = 1$ is forced by monotonicity, lipshitz, and idempotency. Then the behavior on lines y =constant inside the region $0 \leq y \leq x$, is forced by monotonicity. Similarly on horizontal lines when $0 \leq x \leq y$. □

Theorem 9 *If $f(x,y)$ is concave-up (property 11) and agrees with OR at boolean arguments (prop. 4), and if either $f(x,y) \geq \max(x,y)$ (property 19) or f obeys the weakened lipshitz and monotonicity conditions (props. 9, 5) then $f(x,y) = \max(x,y)$. Similarly if concave-down, instead of concave-up, and $f \leq \min$ in place of $f \geq \max$ according to property 19, then $f(x,y) = \min(x,y)$.*

Proof. Concave-$\cup$-ness forces f to lie below or on the lower convex hull of the boolean points $(x=0, y=0, f=0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$, which is $\max(x,y)$. This proves the version in which property 19 is assumed, for OR (the proof for AND=min is exactly similar).

Now if instead of property 19, we are assuming weakened lipshitz and monotonicity, then these assumptions plus the behavior on the lines $x=0$, $x=1$, $y=0$, $y=1$ (forced by monotonicity) suffices to force $f(x,y) \geq \max(x,y)$, that is, property 19 will hold anyway. □

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