On the shape of the universe

Warren D. Smith
WDSmith@fastmail.fm

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Abstract —
1. There are many experimental confirmations of statements that the "stars" (or more precisely, other bright things) at distance $\approx \ell$ from us, are uniformly distributed on the Celestial sphere, provided ℓ is neither too small (to avoid local nonuniformities such as our galaxy) nor too large (to avoid issues of possible "multiple views" of the same star in a non-simply connected universe; so far in practice this has never been an issue). Assume that this angular uniformity is true for almost every observer-location, i.e. that it is not a special consequence of the Earth's location. Also assume that the universe is a boundaryless connected n-manifold (for mathematical purposes we shall allow $n \neq 3$) and that light travels on geodesics. Under these 3 assumptions, we prove a theorem that: if $2 \leq n$ then it is necessary and sufficient that the universe be a "harmonic manifold." It is known that harmonic manifolds are always "Einstein" and that if $n = 3$ these two notions – and constant curvature – all are equivalent. (If $n = 2$ harmonic manifolds and manifolds of constant curvature are the same thing. If $n \geq 4$ then harmonic manifolds exist which are not of constant curvature, and Einstein manifolds exist which are not harmonic.)

2. We argue that the universe must be orientable and cannot contain a geodesic such that traveling along it causes rotation; otherwise known microscopic-scale laws of physics apparently would lead to either contradictions or disagreements with experiment.

3. We review recent (incompletely convincing) experimental evidence that the universe contains a nonzero finite number (e.g., 1) of short closed geodesics passing through the Earth. If that is so, we prove (under our first 3 assumptions, plus our 2 assumptions about orientability) that the candidates for the topology of the universe may be winnowed down to exactly *one* family: the flat 3-torus (parallelipiped with opposite faces identified) or its degenerate versions with some of the parallelipiped sidelengths made infinite.

Keywords — Shape of the universe, Einstein manifold, constant curvature, sectional curvatures, Riemannian geometry.

1 Introduction

Much attention [30][15] has been devoted to the question of what the topology and geometry of the universe (as a 3-manifold) might be. Whenever we use the word "manifold" (including here), we shall mean a boundaryless Riemannian connected manifold. The prefix "n" (e.g. "n-manifold," "n-area," "n-geometry") means "n-dimensional."

This paper will attack the problem by

1. Proving theorems and
2. Considering experimental/statistical evidence of four kinds:
   (a) the observed angular uniformity of various visible objects in the sky,
   (b) orientability properties of the universe apparently required in order for well-confirmed microscopic-scale physical laws to behave palatably,
   (c) recent incompletely convincing evidence that we lie on a closed geodesic within our universe,
   (d) the overwhelming prior likelihood that the Earth does not lie at an atypical "special" location in the universe.

(For precise definitions and statements, see the text.) In particular, our final theorem will show that assuming all of these things together forces the universe to be a 3-torus (parallelipiped with opposite faces identified)[1] or one of its degenerate cases in which 1 or 2 of the parallelipiped sidelengths are infinite.

2 Brief history of previous cosmical models

In the absence of observational constraints, cosmologists have resorted to guidance from inner philosophical desires. For example, Einstein found it philosophically desirable that the universe should be both highly symmetrical and unchanging. This led him to propose certain static cosmical models, involving a nonzero "cosmical constant" modifying the equations of general relativity. Einstein later branded that "the greatest mistake of my life." These models later were abandoned by everybody, including Einstein, because (1) they conflict with the experimental fact that our universe is expanding, and (2) because they are unstable to

[1] For non-mathematical readers: The simplest kind of parallelipiped is a box. A 3-torus is such a box with "wraparound" on all 3 pairs of sides, i.e. going out the top side of the box causes you to come in the bottom side. The conjecture would be that our universe is such a thing and features such wraparound. This paper shows this is the *only* possibility if the universe satisfies certain axioms, which are supported to greater or lesser degree by experiment.

small perturbations. Later, Fred Hoyle found a "steady-state" universe philosophically desirable. Hence, he advocates cosmical models in which matter is continuously created in the vacuum at tiny rates, by means of some unknown new physical process disobeying the conservation of mass-energy[2]. Almost all cosmologists today believe that the experimental evidence disproves this. Both Hoyle and Einstein were affected so strongly by their philosophical desires that they were willing to alter the laws of physics drastically.

The Friedmann-Robertson-Walker models, the most popular ones today ([5] ch.4; [19] ch. 27-29) (and which the present paper will end up supporting) all are based on inner philosophical desires for **homogeneity** (almost every point in the universe should "feel the same" as almost every other point) and **isotropy** (at almost every point, almost all directions should "look the same"). These desires both are, in essence, a desire for more symmetry. E.g. if the universe had a transitive isometry group (i.e. in which some isometry exists mapping each point of the universe to any other) then homogeneity would be guaranteed.

The cartesian product $S^2 \times S^1$ of a 2-sphere and the circle is a (counter)example of a *non*isotropic 3-manifold with a transitive isometry group. But in the other direction, *Schur's theorem* [31][10] shows that any manifold whose Riemann tensor is isotropic everywhere, must have constant sectional curvatures (i.e., in the usual phrasing, "constant curvature")[3].

Nobody seems to be willing to say so explicitly, but the above desires for symmetry may have really been driven more by a desire to keep the models easy to analyse, than by a desire to seek the truth.

Rather peculiarly, though, the symmetry-seeking cosmologists did not try to take the obvious next step, which was to suppose that the universe had a *distance-2-transitive* isometry group – i.e. in which some isometry exists mapping each point-*pair* of the universe to any other point-pair having the same separation. That would automatically yield both homogeneity and isotropy, but also much more. Indeed, the "2-point homogeneous spaces" have been completely classified[4] [35]! The only compact 3-manifolds in Wang's list ([35]; or see theorem 8.12.2 of [37]) are 3D spherical space S^3 and 3D elliptic geometry (i.e., S^3, with antipodal points identified); and the only noncompact ones are Euclidean 3-space $\mathbf{R}^3$ and hyperbolic 3-space $\mathbf{H}^3$.

If we demand compactness, the only Wang-candidates are S^3 and elliptic 3-geometry. Unfortunately, the present observational evidence seems to favor negative [13][20] or zero [14][24][28] curvature and hence would rule out both of these. (Also, the companion paper [29] mitigates against these two.) On the other hand, there is now some observational evidence ([32]; we shall discuss this in §9) suggesting that at the very largest observable length scales, the universe may have a distinguished direction, which perhaps corresponds to a closed geodesic. That would rule out both $\mathbf{R}^3$ and $\mathbf{H}^3$ and indeed could contradict isotropy. If all this is so, then the entirety of Wang's list would be ruled out – in which case, once again, philosophy has failed.

In the other direction (toward less symmetry), one could ask merely that the universe be only *locally* homogeneous, i.e., that a ball of sufficiently small radius ℓ centered at almost every point, is isometric to a radius-ℓ ball centered at almost any other point. Similarly, one could also ask only for *local* isotropy, i.e., that a ball of sufficiently small radius ℓ centered at almost every point, is isometric to all rotated versions of itself. Indeed, these two demands are undoubtably the most popular course at the present time [5][19]. They lead precisely to the 3-manifolds of constant curvature, of which there are an infinite number [37][15], as the candidate topologies for the universe.

But in view of all the historical examples where *philosophy failed* to deliver valid cosmologies, it would be better to base this conclusion on *observational evidence*. That is what the present paper, via new theorems, shall do.

3 The angular uniformity of the sky

One of the few relevant facts astronomers are confident of is that things in the universe look uniformly angularly distributed on the celestial sphere. Examples: The cosmic microwave background (CMB) (once corrections are made for Doppler shifts caused by the absolute motion of the Earth relative to it) is angularly uniform to 10^{-5}, although there are genuine departures from uniformity at the 10^{-5} level [32]. The distribution of the 30821 brightest 4.85GHz radio sources in the sky, and of 681 gamma ray bursts longer than 2 seconds [2][5], and the cosmic X-ray background, all appear to be angularly symmetric to within a few percent[6].

A study involving an amount of data far in excess of all of these is the ongoing Sloan Digital Sky Survey (SDSS). The SDSS confirms previous findings that galaxies tend to be concentrated on clusters, sheets, and filaments, with voids between – all of which constitute departures from precise angular uniformity, leading to approximately power-law angular correlation functions. These effects complicate any argument that the "sky is angularly uniform." On the other hand, proponents of uniformity can content themselves with

[2] Actually, Hoyle could keep energy-conservatiion by introducing a new "creation field." Either way, new physics is required.

[3] There is no direct observational evidence for the isotropy of the Riemann tensor in the neighborhood of our Galactic cluster, because it it not possible to measure the Riemann tensor in that neighborhood without traveling to places millions of light years away.

[4] Incidentally, I am unware of any such classification of a corresponding classification of *Lorentzian* manifolds. Presumably the only eligible $(3+1)$-manifolds would be the de Sitter and anti de Sitter spaces, and flat $(3+1)$-space, all of which have constant curvature.

[5] The 251 bursts shorter than 2 seconds are *not* angularly symmetric [2] with confidence exceeding 99%. Long bursts produce "harder" gamma rays and at least some of them are known (from spectral effects caused by gas between Earth and the burster) to be cosmological distances away. The short bursts presumably also lie at huge distances away (since their angular nonuniformity seems incompatible with what would be expected from assuming they are in our galaxy) but at closer distances than the uniformizing length scale.

[6] The universe's "angular velocity of rotation" is $< 10^{-8}$ radians/year according to planetary observations combined with VLBI quasar observations. Some other calculations (probably less reliable) based on the CMB claim to set upper limits of only 5×10^{-12} radians/year, or even 10^{11} times smaller limits still (surveyed in [5], end of chapter 4).

the knowledge that (1) these correlations *are* power-law-like, i.e. disappear[7] at large distances, and (2) as we go further back in time (larger redshifts), these correlations decrease and uniformity increases [6][9].

4 The non-specialness of the Earth

It seems exceedingly likely that the Earth's position in the universe is in no way special. If so, any observer located almost anywhere else would also see a similarly angularly-uniform sky and similar behavior of the CMB.

5 What can we deduce from angular-uniformity and non-specialness?

An extremely obvious question is: what can we deduce about the shape of the universe [15][30] from this angular uniformity and the assumption the Earth's location is generic? Oddly enough, this mathematical question apparently has not been previously addressed.

These two assumptions may seem very mild, but we shall show that they fully suffice to yield far-reaching conclusions. Namely[8] one concludes that the universe (as a 3-manifold) has constant curvature – agreeing with, and justifying, the most popular present cosmical models (Friedmann-Robertson-Walker; which arose from the desires for local isotropy and local homogeneity).

The proof is not difficult, provided we stand on the shoulders of the giants who created Riemannian geometry. However, one natural n-dimensional version of our theorem, $n \geq 3$, is that the universe should be an *Einstein* n-manifold (i.e. one whose Ricci tensor is a constant multiple of its metric tensor). Einstein manifolds originally arose in the investigations of Einstein into his (later abandoned) static cosmical models. Einsteinity happens to be equivalent (prop. 1.120 of [3]) to having constant curvature when $n \leq 3$. That subtlety indicates that a theorem and proof *are* required – just leaving things as a nowhere stated and nowhere proven "folk theorem" is unacceptable.

One might, more strongly, conjecture that constant curvature is required in *every* dimension n in order to cause the sky to look uniform at *all* orders in ℓ, where ℓ is the "distance to the stars we are observing." But this paper will show that conjecture is wrong. The true situation, in *all* dimensions $n \geq 2$, is that angular sky-uniformity and the nonspecialness of the Earth are *equivalent* to the universe being a "harmonic" manifold. (Harmonic manifolds are discussed in chapter 6 of [36] and pp.370-375 of [23].) These notions (and an additional notion called "super-Einsteinity") obey the following inclusions

$$\text{Const. Curvature} \subseteq \text{Harmonic} \subseteq \text{Super-Einstein}$$
$$\subseteq \text{Einstein} \subseteq \text{All manifolds}. \quad (1)$$

When $n \geq 4$, all these inclusions are strict – except that the smallest known n for which a non-super Einstein n-manifold is presently known, is $n = 7$. However, when $2 \leq n \leq 3$, the first four of these notions all coincide.

6 The uniform-sky theorem and its proof

We shall adhere to the "Einstein summation convention" [5][19] that repeated tensor indices are summed over.

Definition 1 *A Riemannian n-manifold has "constant curvature" if it is locally isometric to (a scaled version of) either $\mathbf{H}^n$ (negative curvature), $\mathbf{R}^n$ (flat), or S^n (positive curvature), i.e. if and only if it is locally isometric to a nonEuclidean (or Euclidean) geometry [4][11][31].*

Definition 2 *A Riemannian n-manifold with Riemann curvature tensor R_{abcd} has "sectional curvature" (in the 2D submanifold defined by two mutually orthonormal directions $\vec{v}$ and $\vec{w}$ at some point) $R_{abcd}v^a w^b w^c v^d$.*

Fact 3 *A Riemannian n-manifold has "constant curvature" if and only if all its sectional curvatures are everywhere equal [25][37][10].*

Definition 4 *A Riemannian n-manifold is "Einstein" [3][25][10] if its Ricci tensor $R_{\mu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa}$ (the Riemann tensor with two indices constracted) is a constant scalar multiple of its metric tensor: $R_{\mu\kappa} = cg_{\mu\kappa}$ for some constant c.*

Definition 5 *The "scalar curvature" K is the full contraction of the Riemann tensor, i.e. the trace of the Ricci tensor, i.e. $K = R_\mu{}^\mu$.*

Definition 6 *An n-manifold is "harmonic" [36] if, for each point $p \in M$, there exists a nonconstant function $F(r)$ of the distance r to p, that is harmonic (i.e. has zero laplacian: $F_{;\mu}^{;\mu} \equiv \nabla_\mu \nabla^\mu F = 0$) on M.*

Remarks about harmonic metrics. The whole topic of harmonic n-metrics is surveyed extensively by Willmore [36]. It is necessary that $F'(r) = k/S(r)$ where $S(r)$ is the surface area of a ball of metrical radius r on M centered at p, and k is a nonzero constant, because $\vec{G} = F'(r)\vec{1}_r = \vec{\nabla} F$ and $F_{;\mu}^{;\mu} = 0$ then occurs if and only if $\vec{G}$ is divergence-free, which by Gauss's divergence theorem (which implies conservation of outward flux of $\vec{G}$) can happen only for *this* choice of $F'(r)$. Willmore ([36] 6.9.1) shows M is harmonic

[7] Peebles [22] claims the mass density in the universe appears uniform to $\lesssim 10^{-4}$ when averaged over volumes corresponding to length scales of $\geq 10^{10}$ light years (10^{26} meters). Statistical analyses [18][16][21][33][34] of the Las Campanas Redshift Survey [27] (resulting in angular and redshift "coordinates" for 26418 galaxies) show that the fluctuations in the luminous mass density in the universe are $\lesssim 5\%$ when averaged over balls $\geq 3 \times 10^8$ light years (3×10^{24} meters) wide. For balls of diameter $\lesssim 3 \times 10^7$ light years, the RMS density fluctuation is as large as the density itself, and for smaller balls it exceeds it.

[8] Starkman [30] mentioned that the CMB is uniform to 1 part in 10^5, and claimed that therefore, the curvature of the universe must be constant to 1 part in 10^4. He did not explain how he made that inference, but this statement suggests that cosmologists had at least conjectured at least some part of our theorem 7. No such theorem is mentioned in the reviews [30] or [15], though.

implies M is Einstein, and indeed, more strongly, implies that M is "k-stein" for every $k = 1, 2, 3, \ldots$. In particular, the next condition beyond Einsteinity is that

$$R^{\alpha\mu\beta\nu} R_{\alpha\theta\beta\phi} x_\mu x_\nu x^\theta x^\phi = c|x^\kappa x_\kappa|^2 \tag{2}$$

for some constant c (there being implied Einstein-summations over repeated indices here) for any vector field x^κ. When $n \in \{2, 3\}$ Willmore [36] shows harmonicity implies Einsteinity. Lichnerowitz and Walker showed in 1945 ([36] 6.9.4) that a *Lorentzian* harmonic manifold must have constant sectional curvature, but for *Riemannian* manifolds this is in general only true when $n \in \{2, 3\}$. Willmore ([36] 6.9.2) shows a harmonic manifold must satisfy

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \text{constant} \tag{3}$$

as well as some other similar conditions. Since not all Einstein 4-manifolds obey EQ 3, an immediate consequence is that non-Einstein harmonic 4-manifolds exist. The complex projective plane CP^2 is an example of a harmonic 4-manifold ([36] table 6.2) which is not of constant curvature. It is known that Einstein and harmonic 4-manifolds are especially nice: harmonic 4-manifolds obey $\nabla_\eta R_{\alpha\beta\mu\nu} = 0$ ([36] 6.10.1) and Einstein 4-manifolds are automatically super-Einstein. Lichnerowicz [17] proved that the harmonic 4-manifolds are precisely the rank-1 symmetric 4-manifolds. Harmonic 4-spaces are also discussed by Saloman [26]. In each dimension $n = 4k+3$ for $k \geq 1$, Gray and Vanhecke [12] used quaternions to construct Einstein n-manifolds which are not super-Einstein. It is known that noncompact unsymmetric harmonic 7-manifolds exist [7].

The following three theorems seem particularly important:

1. ([36] 6.7.4) If M is a harmonic metric, then *any* function G harmonic within an embedded metric ball B centered at x (i.e. such that $G_{;\mu}^{;\mu} = 0$ within this ball), must obey the "spherical averaging principle" that $G(x)$ is the average, over y on ∂B, of $G(y)$.

2. ([36] 6.7.6) If M is a real-analytic manifold such that harmonic functions on M obey the spherical averaging principle, then M has a harmonic metric. (This and the preceding statement together constitute a different characterization of Harmonic manifolds.)

3. ([8]) If M is an Einstein manifold, then M is real-analytic.

Theorem 7 (Uniform-sky⟺Harmonic⟹Einstein) *Let M be a Riemannian n-manifold, $n \geq 2$. Suppose that "stars" (points) are distributed according to some smooth probability density on M. Suppose that generic point-observers on M agree that the stars in a spherical shell of inner radius ℓ and outer radius L centered at his location (for $0 \leq \ell < L$, for all sufficiently small L) are angularly uniformly distributed on his $(n-1)$-sphere "sky." Then:*

1. *The stars are uniformly distributed (equal n-volumes are equally likely to hold a star) .*

2. *If $2 \leq n \leq 3$, then M has constant curvature.*

3. *If $2 \leq n$, then M is Einstein.*

4. *If $2 \leq n$, then M is Harmonic.*

In the other direction to #s 2 and 4: If M has constant curvature or more generally if M is harmonic, then our uniform-sky supposition holds for all observers.

Proof. If the star-density were nonuniform, then an observer located at a place where the gradient of that density were nonzero (and there must be a nonzero measure set of such points), would see a non-uniform sky for all L sufficiently small that curvature effects could be neglected. So from now on assume the star-density is uniform.

The final sentence of the theorem is obvious if L is smaller than the embedding radius of a metric ball in M.

We shall use the fact [4][11] that the area of a circle of radius L in a plane (i.e. 2-manifold) with constant sectional curvature S is

$$\text{area} = \begin{cases} L^2\pi, & \text{if } S = 0 \\ 2\pi|S|^{-1}[1 - \cos(L\sqrt{|S|})], & \text{if } S > 0 \\ 2\pi|S|^{-1}[\cosh(L\sqrt{|S|}) - 1], & \text{if } S < 0. \end{cases} \tag{4}$$

Similarly, the circle's perimeter is

$$\text{perim} = \begin{cases} 2\pi L, & \text{if } S = 0 \\ 2\pi|S|^{-1/2}\sin(L\sqrt{|S|}), & \text{if } S > 0 \\ 2\pi|S|^{-1/2}\sinh(L\sqrt{|S|}), & \text{if } S < 0. \end{cases} \tag{5}$$

If we restrict attention only to an angle-θ wedge of this circle, these area and perimeter formulas of course must be multiplied by $\theta/(2\pi)$.

Proof of constancy of curvature if n=2: If $n = 2$, then suppose (for a contradiction) that M has non-constant curvature S but nevertheless has angularly uniform skies. Then there must exist points on M (in fact a set of such points having nonzero measure) at which the gradient of S is nonzero. An observer at such a point would see fewer stars by looking for a distance L into a small-angle cone in the direction $\vec{\nabla} S$ (for a sufficiently small distance L) than he would see by looking along the antipodal cone. (The fractional increase in stars would be proportional to L^3.) When $n = 2$ this simple argument suffices because the Riemann and Ricci curvature tensors both reduce to scalars. But when $n \geq 3$ they are genuinely tensorial so that a more complicated argument is needed. (For all n, even if $n \geq 3$, the present argument suffices to show that the *scalar* curvature K must be constant.)

Proof of Einsteinity if n=3,4,5...: Now let $n \geq 3$ be fixed. Consider an observer looking in some direction $\vec{v}$ where $|\vec{v}| = 1$. (The vector-arrow notation indicates elements of the tangent space of M.) Let the k-volume of a Euclidean k-ball of unit radius be

$$\bullet_k \equiv \pi^{k/2}/(k/2)!. \tag{6}$$

The surface $(n-1)$-area of a spherical cap consisting of the far endpoints of the length-L geodesic-segments (starting at the observer) with angle $\leq \theta$ to $\vec{v}$, is of course asymptotic to $\bullet_{n-1}(L\theta)^{n-1}$ if $\theta \to 0^+$ and then $L \to 0^+$. But, to a higher order of accuracy in L, it is

$$\bullet_{n-1} \prod_{i=1}^{n-1} \frac{\theta}{\sqrt{|S_i|}} \frac{\sin}{\sinh} (L\sqrt{|S_i|}), \tag{7}$$

where S_i is the sectional curvature in the plane defined by the two directions $\vec{v}$ and $\vec{d_i}$, where the $\vec{d_i}$, and $\vec{v}$, all are mutually orthonormal. (The upper function is used when $S_i > 0$.) By using the Maclaurin series for sin and sinh, this is asymptotic as $L \to 0^+$ to

$$\bullet_{n-1}(L\theta)^{n-1} \left[1 \mp \frac{L^2}{6} \sum_{i=1}^{n-1} S_i\right] \tag{8}$$

As is well known [10], EQ 8 is *independent* of the $\vec{d_i}$ (provided mutual orthogonality holds) and is equal to

$$\bullet_{n-1}(L\theta)^{n-1} \left[1 \mp \frac{L^2}{6} r(\vec{v})\right] \tag{9}$$

where $r(\vec{v}) = R_{\mu\kappa} v^\mu v^\kappa$ is the so-called "Ricci curvature in direction $\vec{v}$."

Incidentally, one might worry that the "orthogonal" planes (2D submanifolds of M) defined by $\vec{v}$ and the $\vec{d_i}$ will "twist" or "shear" *away* from being orthogonal as we walk along the geodesic in the direction $\vec{v}$, requiring additional correction terms in EQ 8. This fear is justified, but irrelevant since those correction terms will involve higher powers of L since they depend on derivatives of R_{abcd} rather than on the curvature R_{abcd} itself.

We now see that an observer's sky-uniformity must be violated (with the fractional increase in stars being proportional to L^2) unless $r(\vec{v})$ is independent of $\vec{v}$, for that observer. I.e., unless the $(n-1)$-area squashing factor (EQ 9) is independent of the direction $\vec{v}$ in which we gaze, the apparent star-density will differ in different directions. But this $\vec{v}$-independence condition, as is well known [3][10], is equivalent to requiring that M be an Einstein manifold.

Remarks about Einsteinity: If $2 \leq n \leq 3$, then Einsteinity and constant curvature are the same thing (prop. 1.120 of [3]). Thus much of our theorem could have been more simply stated, for all $n \geq 2$, as "M is Einstein." If $n \geq 3$, Einstein manifolds have constant *scalar* curvature [10], but the reverse is not the case since $S^2 \times S^1$ has constant scalar curvature but is not Einstein. If $n \geq 4$, Einstein n-manifolds of nonconstant curvature exist; an example is the complex projective plane CP^2.

Harmonicity implies uniform skies if n=2,3,4,...: Consider the function $F = k/S(r)$ where r is the distance to the point-observer p, k is a nonzero constant, and $S(r)$ is the surface area of the sphere of radius r on M centered at p. Obviously the geodesics emanating from p go, everywhere, in the direction $\vec{\nabla} F$. Due to the harmonicity of F, i.e. the zero-divergence of $\vec{\nabla} F$, these "lines of flux" are conserved (except at p itself). Hence if p sees a uniform sky (consider a very small, hence approximately Euclidean, "sky" sphere centered at p) then (by following the geodesics) the stars on any sphere of any radius r centered at p (or on any thin spherical shell from r to $r + \mathrm{d}r$) must be uniform (since F, and hence $\vec{\nabla} F$, depend only on r).

Proof uniform skies imply harmonicity if n=2,3,4,...: If observer p sees uniformly starry skies, then again consider F and the radial geodesics going in the direction $\vec{\nabla} F$. Were M not harmonic then the divergence of $\vec{\nabla} F$ would (for a nonzero measure set of locations and a nonzero measure set of p) necessarily be nonzero – and in a manner *not* spherically symmetric about p. Because: if it were symmetric, then the conservation of "flux" of $\vec{\nabla} F$ would by the divergence theorem imply zero divergence everywhere, which would contradict the assumed non-harmonicity of M. (Indeed this indicates that this divergence must be in some places positive and in some negative so that it can average to 0 within p-centered balls.) It is easy to see that this would lead to non-uniformly starry skies as viewed from p. This contradiction forces the conclusion that F is harmonic. Q.E.D.

7 General relativity

Our argument has not invoked general relativity (GR), aside from assuming the universe is a 3-manifold and that light travels on geodesics.

However, GR really describes the universe as a $(3+1)$-dimensional *Lorentzian* manifold, a fact we've disregarded. That is OK. Time evolution via GR of a 3-manifold universe of constant curvature, filled with matter, stars, and/or isotropic radiation, at constant density, will yield at all future times, a universe which also has these same curvature and density properties. (Because: GR is a purely local set of PDEs...) In fact this is precisely the Friedmann-Robertson-Walker model. Given equations of state for the matter and radiation, GR will wholy determine the (time-evolution) behavior of the metric for us [5][19]. So: the present paper's decision to restrict attention to purely-spatial ("Riemannian") manifolds presumably lost nothing.

8 Orientability: 3 assumptions

1. The experimentally observed parity non-invariance of the weak force, and the experimentally observed unidirectionality of time, both strongly suggest that the universe must be an *orientable* 3-manifold.

This has been previously remarked [15]. But we can say more.

2. Suppose the universe contains a nonorientable 2D submanifold. For example: elliptic 3-geometry is orientable, but it contains elliptic 2-geometry, which is non-orientable. In that case, an electron confined by some potential "walls" to lie on or near this 2D submanifold, and which had "spin-↑" (that is, pointing perpendicularly to the submanifold) would be able, by going "around the universe," to convert to having "spin-↓." In that case, the Dirac equation could have no solution corresponding to an electron "spin-↑ traveling wave" (simultaneous spin and momentum eigenstate) and would only be able to have solutions corresponding to a "superposition of spin-↑ and spin-↓." This seems contrary to experimental reality. This weakly suggests that the universe *cannot contain a non-orientable 2D submanifold.* (I say "weakly" because it seems impossible to hope for genuine experimental confirmation of this idea; we can only perform this "gedanken experiment.")

3. Next, consider an infinite, constant-width slab in Euclidean 3-space, e.g. $\{(x, y, z)$ such that $|z| < 1\}$, with the upper face ($z = 1$) of this slab identified with the lower face ($z = -1$) via a screw-translation, with screw-angle θ, along the z-axis. In such a universe, the z-axis would be a closed geodesic. An electron with "spin-↑" (*perpendicular* to the z-axis) would, upon traveling along the z-axis, have its spin repeatedly rotated though an angle θ. If $\theta \neq 0$, that would cause our spin-↑ electron to convert (with nonzero probability) to "spin-↓" despite never interacting with anything else. Again this seems contrary to experimental reality. This weakly suggests that the universe *cannot contain a closed geodesic such that traversal along it (returning to the starting point) causes rotation.*

Both of the above "weakly" supported orientability assumptions may also be inferred in a different way, not involving electrons with spin, but instead involving photons with polarization. Thus, in the last case, a photon traversing the closed geodesic many times will cancel itself out because its direction of polarization would be rotated and hence would sum to $\vec{0}$. Thus, it would be impossible, in such a universe, to have a "photon in a polarization & momentum simultaneous eigenstate" moving along that closed geodesic. In contrast, the previous argument showed the impossibility of having an "spin-↑ electron in a momentum eigenstate" moving along that closed geodesic.

We also **remark** that our 2$^{\text{nd}}$ assumption (about 2D submanifolds) will not be needed in the rest of this paper, since:

Theorem 8 (3$^{\text{rd}}$ assumption$\Longrightarrow$ 2$^{\text{nd}}$) *The 2$^{\text{nd}}$ assumption of this section (about 2D submanifolds) is a consequence the 3$^{\text{rd}}$ (about closed geodesics).*

Proof: In any non-orientable 2D submanifold of a 3-manifold, consider some orientation-reversing geodesic from a point p to itself – that has rotation $\theta = 180° \neq 0$. The shortest such geodesic is necessarily a closed geodesic. Q.E.D.

9 Recent experimental evidence that may suggest we are on a short closed geodesic of the universe

A recent highly publicized paper by Tegmark et al [32] gave evidence suggesting that the universe contains a fairly short closed geodesic (short enough that it may be within our viewing horizon), or at least, that the universe contains a preferred direction. What follows will be partially a critical review of that paper, and partially an extension (new claims and new deductions) of it.

The automated WMAP observatory (Wilkinson Microwave Anistropy Probe, located at the Lagrange point L_2 in solar orbit 1.5 million km from Earth) measured the cosmic microwave background, making a spherical map containing 3,145,728 "pixels," each pixel having 5 intensity values (at 5 microwave frequencies from 22.8 to 93.5 GHz).

Tegmark et al [32] attempted to remove the unwanted "foreground" from this map so that we would see only the wanted microwave background using statistical methods we will not discuss. After they did it they had a spherical sky map containing 3,145,728 pixels, each coming with a microwave "temperature" value.

Now by integrating this map times $Y_{\ell m}$'s (orthonormal spherical harmonics) for $-\ell \leq m \leq \ell$, one can get:

- from Y_{00}: an "average temperature" value ($\approx$ 2.726°Kelvin);
- from $Y_{1,-1}, Y_{10}, Y_{11}$: 3 "dipole" coefficients;
- from $Y_{2,-2}, Y_{2,-1}, Y_{20}, Y_{21}, Y_{22}$: 5 "quadrupole" coefficients;
- from $Y_{3,m}$ for $m = -3, \ldots, 3$: 7 "octupole" coefficients;
- et cetera.

One might expect a priori that the sum of the $2\ell + 1$ absolute squares of the ℓ-pole coefficients (where $\ell = 2$ for the quadrupole and $\ell = 3$ for the octupole case, etc.) would be, if appropriately normalized, a random variable with a $(2\ell + 1)$-degree-of-freedom-chi-squared distribution. (Note: A "chi-squared variable with n degrees of freedom" is the sum of the squares of n standard normal deviates. It has density

$$x^{(n-2)/2} e^{-x/2} 2^{-n/2} / \Gamma(n/2) \quad \text{for} \quad x \geq 0. \tag{10}$$

mean = n, and variance = $2n$. Note also that since we have complex coefficients one might expect twice as many degrees of freedom, but really due to complex conjugation symmetries when $m \to -m$ this doubling does not happen.)

But Tegmark et al (as well as other previous investigators) discovered that the sum of the 5 squares of the quadrupole coefficients is anomalously low. In fact, the probability that such a chi-squared variable would be that low (or lower) was only 5.1% according to Tegmark et al (other values between 1.8% and 5.1% had been given by other investigators).

While that is suspicious, 95% significance by itself isn't conclusive. However, Tegmark et al then noted that the 5 quadrupole terms had[9] anomalously low power in the 2 directions of a certain axis in space. The octupole terms *also* had anomalously low power on that *same* axis in space, and in particular there is a certain direction[10] along which there is near-minimal quadrupole *and* octupole power (and, we might add, hexadecapole power, and indeed overall power – according to their figure 1 – also are low there). Tegmark et

[9] If plotted on the sky map combined, but with all the other ℓ-pole terms removed; see their figure 15 which needs to be viewed in color, and also it is unfortunate they did not give latitude and longitude markings.

[10] Note: I am now *not* speaking of a bidirectional axis, but a unidirectional direction – it is slightly to the right of and below the center of their oval sky map and lies on an Earth-Virgo line

al unfortunately did not explicitly compute the likelihood of this quadrupole-octupole direction coincidence, but our own rough computation[11], indicates approximately a 1-in-5 chance, and if one considers the hexadecapole power-low also, then that lowers this to a 1-in-20 chance.

That *combined* with that 5.1% likelihood brings us down to 0.25% fluke-chance... i.e. 99.75% significance level, that there is a visibly distinguished direction in the universe, such as a short closed geodesic[12]. More precisely, this 99.75% significance supports any hypothesis whatever that would predict *both* low overall quadrupole power *and* a distinguished low-power direction, versus the null hypothesis. *That* significance level *is* worth getting excited about.

10 Allowable kinds of universes

Suppose that the universe really does have some topology containing a short closed geodesic through the Earth. Using our assumption of the non-specialness of the Earth's location, we conclude that some short closed geodesic passes through almost every point in (i.e. a nonzero-measure subset of) the universe. Furthermore, if there is exactly *one* closed geodesic (or some nonzero finite number F of closed geodesics) passing through the Earth (regarded as a point), we similarly conclude that exactly F short closed geodesics pass through almost every point in the universe. Furthermore, assume the universe is realized in a way which has (approximately) constant curvature everywhere, due to theorem 7. Then we can winnow down the possible candidate topologies for the universe:

Theorem 9 (Closed geodesics) *Let M be an n-manifold, $n \geq 2$, with constant curvature. Suppose M has a minimum-length closed geodesic through every point (or at least through a nonzero-measure subset of its points). Then M is either n-dimensional elliptic geometry, the n-sphere S^n, or a flat n-dimensional manifold. If only a finite nonzero (or at most countably infinite) number of closed geodesics pass through each point, then only the lattermost of these 3 possibilities is allowable.*

Proof. In order for M to have constant curvature it must be gotten by gluing polytopes (all the polytopes are cut from just one of the three geometries S^n, $\mathbf{H}^n$, or $\mathbf{R}^n$) together at common faces. (Infinite "polytopes" are permitted.)

Now in anything other than the cases mentioned, such glued polytopes will necessarily have a finite number of minimum-length closed geodesics (or perhaps countably infinite, if a countably infinite number of polytopes are glued) – i.e., almost all points will not lie on *any* closed geodesic (especially not one of minimum length). That is because for each possible combinatorial type ("symbolic dynamics") of geodesics vis-a-vis the polytope faces, we may perform a minimization ("pull the string taut"). This minimization cannot yield a continuum-infinity of solutions. In fact it cannot even yield *two* solutions. The reason is that, if there were two equi-minimal geodesics of the same combinatorial type, then there would be, in the covering space M^* (which is S^n, $\mathbf{H}^n$, or $\mathbf{R}^n$) of our manifold M, a plane quadrilateral whose 4 angles summed to 360°. Two opposite sides of this quadrilateral are our geodesic segments, and the other two sides lie on two polytope face-hyperplanes (which are really opposite sides of the *same* hyperplane in M, although in M^* this is not true). But that is impossible [4][11] *unless* space is flat ($M^* = \mathbf{R}^n$), or *unless* there are no facial hyperplanes ($M = S^n$) or *unless* there is only one hyperplane involved and the geodesic forms a 2-gon in M with the hyperplane (which, in M^*, is not a polygon at all), which can only happen if M is elliptic n-geometry.

Finally, observe than on S^n, any geodesic emanating from a point, returns to that point length 2π later, i.e. every geodesic is an equi-minimal closed geodesic. In elliptic n-geometry, any geodesic emanating from a point, returns to that point length π later (this is best seen by pre-performing an isometry of elliptic space to place the point under consideration at the "center"), i.e. again every geodesic is an equi-minimal closed geodesic. Hence, if it is *not* true that an uncountably infinite number of minimal geodesics pass though a point, we can rule out S^n and elliptic n-geometry and conclude that M must be flat. Q.E.D.

Remark. In theorem 9 it is not necessary to demand that the closed geodesic be "minimum length," because allowing all other closed geodesics will multiply their number by at most a countable infinity (at least, if our manifold is tessalated by a finite number of polytopes), in which case the set consisting of all points in all these geodesics, will still have measure 0.

We may now winnow further:

Theorem 10 (Universe candidates) *Let M be an orientable constant-curvature n-manifold, $n \in \{2, 3\}$, obeying the orientability assumptions of §8. Let M contain a nonzero number (but not an uncountably infinite number) of minimal-length closed geodesics through each point. Then M is an n-torus (parallelipiped with opposite faces identified) or one of its k degenerate variants with k of the n parallelipiped sidelengths made infinite, $0 \leq k < n$.*

Proof. By theorem 9, M must be everywhere-flat.

Let us first handle the (comparatively easy) 2D case $n = 2$. In the 19th century A.F.Möbius [1] completely classified the topological types of 2-manifolds. Specifically, any

[11] To do this: randomly rotate the octupole-terms-only intensity function versus the quadrupole-terms-only function and count the percentage of the time this results in a low-power-direction "coincidence" of at least as high quality as the one they observed. Had Tegmark et al [32] provided the numerical values of the 5 coefficients of the $Y_{2,m}$ and the 7 coefficients of the $Y_{3,m}$, then other people would be able to carry out this computation to high accuracy. Since they did not, we did it crudely by a careful manual examination of their intensity plot in their fig. 15 (taking advantage of the area-preserving property of their Mollweide projection). E.g. from this we conclude that about 3.1% of the octupole, and about 4.4% of the quadupole, intensity pixels have the lowest level of intensity in their respective plots.

[12] It is not evident to me exactly why looking along a short closed geodesic necessarily should yield anomalously *low* CMB power. That depends on models of fluctuations in the plasma that filled the early universe and was the source of the CMB; I am not competent to comment on that. Certainly it seems plausible that compact universes might generate smaller quadrupole terms than noncompact ones.

compact 2-manifold is diffeomorphic either to a sphere S^2 with g handles attached (for some $g \geq 0$) or to S^2 with c orientability-destroying crosscaps added (for some $c \geq 0$). [If g and c are both nonzero then one may replace (g, c) by $(0, 2g + c)$.] From this we may conclude (cf. figure 13 page 155 of [15] and theorem 2.5.5 page 77 of [37]) that the only orientable flat 2-manifolds are the 2-torus, $\mathbf{R}^2$, and the infinitely-long cylinder, and the only non-orientable flat ones are the Klein bottle and the infinitely-wide Möbius strip. Upon discarding the non-orientable ones and $\mathbf{R}^2$ (since it contains no closed geodesics) our theorem follows.

Now let $n = 3$. Wolf [37] in his chapter 3 classifies the flat 3-manifolds into 18 parameterized families.

His key theorem (corollary 1.9.6 of [37]) is that the flat n-manifolds necessarily are the fundamental polytopal regions of an n-dimensional discontinuous group of affine-linear operations.

So Wolf specifies each of his 18 families by giving a finite set of group-generators (each a linear operation mapping $\mathbf{R}^3 \to \mathbf{R}^3$) each of which leave a point on the manifold invariant, and by specifying a set of relations, obeyed by those generators, which are necessary and sufficient to make the manifold *be* a manifold. For example, Wolf's manifold A_2 is an infinite slab, and each point on the top face of the slab is identified with a point on the bottom face got by a screw-translation. (Parameters are slab thickness and screw-angle.)

Let us now discuss Wolf's classification in more detail. Wolf's 18 classes are (from the summary table on page 123 of [37])

- 4 noncompact orientable manifolds, which we will call A_1, A_2, A_3, A_4.
- 6 compact orientable manifolds, which both Wolf and we call G_1, G_2,..., G_6.
- 4 noncompact nonorientable manifolds, which we will call C_1, C_2, C_3, C_4.
- 4 compact nonorientable manifolds, which both Wolf and we call B_1, B_2, B_3, B_4.

Of course, from now on we shall only be interested in the 10 orientable cases A_k and G_k.

The four A's (see Wolf's theorem 3.5.1, [37] page 112):
A_1: $\mathbf{R}^3$. (Has no closed geodesics.)
$A_2(\theta)$: Slab, generator is a screw-translation with screw-angle θ. If θ is a submultiple of 180°, then A_2 contains a nonorientable 2D submanifold (infinitely wide Möbius strip). If $\theta = 0$, A_2 is an infinitized case of G_1.
A_3: Infinitely-long prism (with parallelogram cross section): generators are two translations. (Infinitized case of G_1.)
A_4: Same as A_3 except the top (partially finite) face of the prism is identified with the (same shaped) bottom face via a *central reflection* (2 sign changes) instead of a translation (no sign changes). I.e., the generators are a translation and a 180° screw-translation. Contains a nonorientable 2D submanifold (Klein bottle).

The six G's (see Wolf's theorem 3.5.5, [37] page 117):
G_1: 3-torus (parallelipiped with opposite faces identified) generated by 3 translations.
G_2 though G_5 each are generated by 3 translations *plus* an extra group-operation α which obeys certain magical relationships with the 3 translations. For example in G_2, G_3, G_4 and G_5 respectively, α^k is one of the translations for $k = 2, 3, 4, 6$ respectively (and G_2, G_4, and G_5 each must contain a non-orientable 2D submanifold invariant under α, α^2, and α^3 respectively).
G_3 arises from the 2-manifold corresponding to the tesselation of the plane by regular hexagons, cartesian producted with $\mathbf{R}$ (i.e. it is an infinitely-long regular-hexagonal prism, with certain face-identifications). It contains a 1D submanifold invariant under both α and α^2, on which a traveling electron will be rotated by 120° degrees, and hence (if it has "spin-$\uparrow$") will with good probability be converted to "spin-$\downarrow$." This seems unphysical, see §8.
G_6 is generated by 3 translations plus *three* extra group-operations α, β, γ which obey certain magical relationships with the 3 translations, including that α^2 is one of the translations. Thus again G_6 must contain a non-orientable 2D submanifold invariant under α and hence is unphysical in the sense that it disobeys the orientability assumptionjs of §8.

In conclusion, among the 18 types of flat connected 3-manifolds, there are 10 orientable types, which we denote G_k, $k = 1, \ldots, 6$ (compact; this naming agrees with Wolf's) and A_k, $k = 1, \ldots, 4$ (noncompact; our naming disagrees with Wolf's). Among these 10, we rule out 5 as unphysical since they contain a nonorientable 2D submanifold. Among the remaining 5 types, G_3 and $A_2(\theta)$ if $\theta \neq 0$ both seem unphysical since they contain a 1D submanifold (i.e., closed geodesic) on which a traveling electron with spin-$\uparrow$ (perpendicular to that geodesic) keeps rotating about that geodesic. In other words, these disobey the orientability assumptions of §8.

We now have winnowed Wolf's 18 classes down to just 4 "physically acceptable" types, namely A_3, G_1, $A_2(0)$, and $A_1 = \mathbf{R}^3$. All 4 of these types are really just one type, namely the 3-torus G_1 (a parallelipiped with opposite faces identified), provided we allow limiting cases where 1, 2, or all 3 of the parallelipiped sidelengths become infinite.

Finally, if there is a closed geodesic in the universe, we may discard $A_1 = \mathbf{R}^3$. Q.E.D.

Remarks. Wolf [37] remarks that the compact flat 4-manifolds were (in 1972) classified into 75 types independently by Charlap & Sah, Calabi, and Levine. (But apparently none of them published.) In chapters 4-7, Wolf also presents a complete classification of n-manifolds with constant *positive* curvature.

There are an infinite number of classes of manifolds with constant negative curvature, and they remain unclassified.

Theorem 11 (Alternative) *Let M be an orientable constant-curvature n-manifold, $n \in \{2, 3\}$, obeying the 3 orientability assumptions of §8. Let M contain an uncountably infinite number of minimal-length closed geodesics through each point (or, more weakly, merely through a nonzero-measure subset of the points). Then M is S^3.*

Proof. Immediate consequence of previous 2 theorems, and ruling out elliptic 3-geometry using §8. Q.E.D.

11 Conclusion

The WMAP evidence discussed in §9 suggests (incompletely convincingly) that exactly one minimum-length closed geodesic *does* pass through the solar system.

If so, we could then conclude from the evidence of star-sky angular-uniformity, the assumption of the non-specialness of the Earth's location, and the orientability assumptions in §8 – via theorems 7, 9, and 10 which successively winnow down the possible candidate 3-manifolds – that the universe's topology must be a 3-torus or one of its two partially-infinite variants.

The companion paper [29] had previously given an explanation for the quantization of charge arising from the assumption the universe was a 3-manifold containing both a noncontractible closed curve, and a "topologically trapped magnetic field." That also may be regarded as evidence supporting notion that the universe is non-simply connected and has a closed geodesic. Thus the two papers [29][32] reinforce one another. We [29] also points out that if the universe had *two* homotopically inequivalent uncontractible closed curves, then generically electric charge would be impossible, since there would then be two charge quanta whose quotient would generically be an irrational number. That may be regarded as evidence the universe is a 3-torus.

Our final logical step, i.e. combining theorem 10 with the WMAP evidence about *exactly one* closed geodesic, may be replaced with an alternative argument. One may rely instead on other experimental evidence [13][14][20][24][28] (and, if desired, one could also utilize my own charge quantization argument [29], which breaks down for S^3 and elliptic 3-geometry) against the two positive-curvature universes still allowed by theorem 9.

To apply theorem 9 it is necessary to have evidence for, or belief in, a closed geodesic passing through the Earth. In the absence of such evidence or beliefs, we still have theorem 7 showing the universe must be a 3-manifold with *constant curvature*.

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