Zeropoint energies, cosmical constant, roughness, feedback, and supersymmetry

Warren D. Smith*
wds@research.nj.nec.com

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Abstract — **We explain how the "zero point energies" of vacuum quantum field modes lead to an immense contradiction called the "cosmical constant crisis," then indicate how "supersymmetry" may resolve this crisis harmlessly. All the just-mentioned ideas are 15-70 years old and had known fatal flaws. We then go further and explain why even "broken" supersymmetry combined with some additional ideas about "feedback" and "roughness" should do the job while overcoming the fatal flaws. Only fairly elementary reasoning is required. Although certainly details remain to be worked out, this appears to allow a resolution of one of the biggest mysteries in physics with the aid of one of the most promising ideas in theoretical physics – supersymmetry. This is hopefully just the first installment in a series of papers by me (which will appear at random times and in random order) about a new picture of Gravity. This one should be readable independently of the others.**

Keywords — Supersymmetry, cosmical constant crisis, ultra high energy cosmic rays, zeropoint vacuum energies, spectral roughness, quantum chaos, Anderson localization.

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*NECI, Princeton NJ 08544.

1 Zero point energies, the cosmical constant "crisis," and its new possible resolution via "supersymmetry"

The spacetime vacuum is filled with "zero point modes" (eigenmodes of the various kinds of wave equations satisfied by quantum fields) up to some high-frequency cutoff.

Why? Each eigenmode of a field whose energy density is proportional to the square of the amplitude of that mode (e.g. for electromagnetic field modes, energy density is proportional to the squares of the electric and magnetic fields) acts, mathematically, like a "simple harmonic oscillator," i.e. a quantum system with Hamiltonian $(p^2+q^2\omega^2)/2$ where p is an abstract "canonical momentum" operator, q is a "canonical position"operator, and ω is the angular frequency of that eigenmode. The energy eigenvalues of such a simple harmonic oscillator are (as is well known) $(n+\frac{1}{2})\hbar\omega$ where n is a nonnegative integer, interpreted, in this context, as the "number of particles" (here photons) in that eigenstate. *Note the* $\frac{1}{2}$. Thus, note, modes with *zero* particles in them *still* have energy $\hbar\omega/2$, called the "zero point energy."

Zero point energy effects [43] have been observed experimentally. The two most direct manifestations are the "Casimir effect" and the "Lamb shift." The Casimir effect arises because, in the vacuum between two (clean and flat) metal plates, low frequency EM[1] modes (of wavelength larger than the gap between the plates) cannot exist. Hence, the plates are "squeezed together" from outside by vacuum modes at all frequencies. They are "pushed apart" by vacuum modes at all frequencies *except* for the low ones. Hence there is a net force (the "Casimir effect") pushing the plates together. The closer the plates get, the more modes are excluded and the stronger is the force. This effect was predicted and calculated by Casimir and measured by Sparnaay [60] and later investigators [63][62][55]. In the latest and most precise experiments [55][31] the measured forces agree with the theoretical predictions to $\approx 1\%$.

The "Lamb shift" [69] arises because the modes in the vacuum with occupation number $n=0$ but nonzero energies $\hbar\omega/2$, and hence nonzero electric and magnetic field amplitudes, can shove electrons around like a fluctuating force. These are called "vacuum fluctuations[2]." The vacuum fluctuations shove the electron in a hydrogen atom (or hydrogen-like ion) around, causing its probability distribution to be slightly "wider" than the one you get from the exact solution of Dirac's wavefunction equation for the Hydrogen atom. This shifts the spectral lines slightly away from the Dirac predictions. The effect was both calculated and measured by Willis Lamb, who was awarded a Nobel prize for it. It is also possible [45] to regard the Lamb shift as simply the "Stark shift" of hydrogen's spectrum in an applied field, in the limit when the energy density of the modes of the applied field is $\hbar\omega/2$. The latest theoretical predictions and experimental measurements agree [47][70][18] to within amounts comparable to the experimental error, which varies from 11 parts per million for the 14042.52MHz Lamb $(2S_{1/2}-2P_{1/2})$-shift in hydrogenic helium [70] to 4% for the $1S$-shift (470eV) in hydrogenic uranium [7]. Perhaps the most impressive such experiment for our purposes is the Lamb $(2S_{1/2}-2P_{1/2})$-shift in Li-like Uranium ions, for which the theoretical prediction [47] is $280.54\pm.15$eV, whereas the experimental measurement [57] is $280.59\pm.09$eV. This agreement suggests that QED's notion of vacuum mode zeropoint energies ought to be correct up to 1KeV.

The cosmical constant "crisis" [68][3] is the puzzle that these zeropoint modes, since they are real, ought to weigh.

For example, if one sums the $\hbar\omega/2$'s of all the normal modes of a box full of vacuum up to the Kth mode:

1. If we cut off modes at the[4] "Planck energy" because

[1] We will employ these abbreviations: EM=electromagnetic, GR=general relativity, QCD=quantum chromodynamics, QED=quantum electrodynamics.

[2] You cannot extract energy from these modes, which are already in the ground state. The Casimir effect extracts energy, admittedly, but only by *eliminating* certain modes, thus garnering their ground state energies.

[3] Gerard t'Hooft remarked in a May 2001 public lecture that the fact that the cosmical constant was zero, or small but nonzero, was surely "one of the biggest mysteries in physics." Furthermore t'Hooft said he had often considered writing a paper entitled "200 wrong theories about the cosmical constant," which was planned to have 200 references. I am not t'Hooft and so I will not present a detailed review of other people's attempts to resolve the cosmical constant crisis (for that see [68][56][15]). But I will remark that these attempts all seem to require dubious reasoning, and/or merely transform the puzzle into a new puzzle rather than resolving it. For example, some attempts have involved "anthropic reasoning" (if the cosmical constant Λ were much different from its actual value, we would not be here to observe it). Anthropism makes many people very unhappy – they would prefer basing arguments on something more solid. Other attempts have involved modeling the creation of the universe, including the unknown physical laws involved in that creation. Another idea (tracing back to Einstein himself) was to regard Λ not as a new constant of physics, but rather merely as a "Lagrange multiplier" which (in a certain variational reformulation of GR) we need to choose to match the "boundary conditions." That would merely transform the puzzle into a new puzzle of what the correct boundary conditions for GR are (especially when one is trying to treat the entire universe). Another interesting idea was expounded by Cornelius Lanczos (1893-1974) in a huge sequence of papers [40]; it involves actually having enormous (and attractive) Λ, which causes spacetime to be "crinkly" on microscopic length scales. But then, hopefully, on macroscopic scales spacetime *appears* to be smoothed out and obeying GR with $\Lambda\approx 0$. Unfortunately I have uncovered several key and irreparable flaws in Lanczos's papers which invalidate almost everything in them (I may publish them elsewhere as part of a discussion of Lanczos's gravity work).

[4] Max Planck noticed in 1899 that a complete system of units could be developed from his $\hbar\approx 1.054576\times 10^{-34}$ Joule second, the speed of light $c=2.99792458\times 10^{8}$ meter/second, and Newton's gravitational constant $G\approx 6.673\times 10^{-11}$ meter^{3}kg^{-1}sec^{-2}. Thus we get the "Planck length" $\ell_{\text{planck}}=\sqrt{\hbar G/c^3}\approx 1.6161\times 10^{-35}$ meter, the "Planck mass" $m_{\text{planck}}=\sqrt{\hbar c/G}\approx 21.77\mu$g, the Planck energy $E_{\text{planck}}=m_{\text{planck}}c^2$, the Planck momentum $p_{\text{planck}}=m_{\text{planck}}c$, etc. I will discuss the great importance of Planck units and how and why waves must be cut off roughly at the Planck wavelength elsewhere in this series of papers. For now I content myself with the claims that (1) any attempt to make a wavepacket with wavelength smaller than ℓ_{planck} would necessarily not yield a "traveling wave" at all, but rather a "black hole;"

we assume our quantum theories and general relativity are roughly valid up to that energy, we get a vacuum mass-energy density of 2×10^{92}kg/meter^3.

2. Even if we cut off modes at typical energy scales (250GeV) arising in quantum chromodynamics (since we assume QCD is roughly valid[5] then we still[6] expect a density of 10^{27}kg/meter^3.

3. If we decide not to trust QCD (since, e.g., its mass and magnetic moment predictions have been typically only $\approx 10\%$ accurate so far [1]) and hence take the mode cutoff energy to be only 1KeV (relying now on rock solid experimental evidence of the Lamb shift in atoms and ions) then still the mass density of the vacuum would be 10^{-8}kg/meter^3, which would still be 18 orders of magnitude larger than the observed mass density of the universe.

Thus the vacuum should weigh a tremendous amount, causing huge gravitational effects – none of which are observed![7]. Put another way, the "cosmical constant" Λ [46] ought to be tremendously large, but in fact $|\Lambda|$ is extremely small [2]. It is difficult to explain such a discrepancy of 120 orders of magnitude (or at least 18 or 50) with its roots in such fundamental physics. This is surely the biggest numerical contradiction in physics; it is no wonder Weinberg [68] calls it a "crisis."

This paper examines this issue more closely. First we shall discover that the hypothesis of "supersymmetry" in particle physics actually may provide an explanation for why $\Lambda \approx 0$ (or at least, it should reduce the gap by a large number of powers of 10)! This idea was thought of before [71], but we'll see in §3-5 that one gets twice as many powers of 10 as was previously thought. But this is still not enough. Second, we'll recognize in §13 that a new "feedback" mechanism must be operating, and in the presence of a new phenomenon I call "spectral roughness," which should get the rest of the way. The observed fact that $\Lambda \approx 0$ now may be regarded as experimental support for supersymmetry.

(2) An entropy bound by Bekenstein [5] rules out the possibility of storing ≥ 1 bits of information in a container whose surface area is $\leq ???(\ell_{\text{planck}})^2$, whereas if a photon could be contained within a box of surface area $\leq ???(\ell_{\text{planck}})^2$ that (by virtue of its 2 possible polarization states) would be 1 such bit. The most important thing to notice about m_{planck} is that it is over 10^{19} times the proton mass; similarly ℓ_{planck} is over 10^{19} times smaller than atomic nuclei. Hence it is often very appropriate to employ asymptotic methods when dealing with Planck lengths or masses.

[5] Also, since multijoule particle energies have been observed in cosmic rays [9], it would be peculiar if modes this high energy can be occupied, but could not be unoccupied.

[6] For comparison, the mass of the planet Earth is only 6×10^{24}kg. Another interesting comparison: if the present mass of the universe were placed in a Euclidean ball of radius r, then that mass would exceed the mass due to zero point modes (with Planck cutoff) only if $r \lesssim 10^{-13}$ meter, i.e. about 30 times larger than the "classical electron radius."

[7] In order to be roughly consistent with known bounds on the cosmical constant (which are of the same order as the – very small – mass density of the universe), the energy cutoff would have to be of order.03eV. Coincidentally (?) this is about the same as the known experimental lower bound on the largest neutrino mass.

2 Creation and Destruction operator formalism and zeropoint energies

This section is based on Dirac's lectures [17].

It is possible to rewrite the simple harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2\omega^2), \tag{1}$$

where $qp - pq = i\hbar$, as

$$H = (C\overline{C} + \frac{1}{2})\hbar \tag{2}$$

where

$$C = \frac{p + iq\omega}{\sqrt{2\hbar}}, \quad \overline{C} = \frac{p - iq\omega}{\sqrt{2\hbar}} \tag{3}$$

are called the "creation and destruction" operators in the context of quantum field theory. (In the context of the harmonic oscillator outside of quantum field theory, they are usually called "raising and lowering" operators.) They obey

$$HC - CH = C\hbar, \quad \overline{C}C - C\overline{C} = 1. \tag{4}$$

Now if there exists a state $|0\rangle$ such that $\overline{C}|0\rangle = 0$, then $|0\rangle$ is unique. Then for every integer $n \geq 0$ we may define the state $|n\rangle = C^n|0\rangle$. The full space of states spanned these as a basis is called the "Fock space." One may show

$$H|n\rangle = (C\overline{C} + \frac{1}{2})\hbar\omega|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle \tag{5}$$

so that the energy of $|n\rangle$ is $(n + \frac{1}{2})\hbar\omega$. We interpret n as the "number of particles" and $|n\rangle$ as the state corresponding to "occupation (of that mode) by n particles." (In the context of the harmonic oscillator outside of quantum field theory this is just the nth excitation level of the oscillator.) $N = C\overline{C}$ is the "number of particles" operator. The reason C is called a "creation" operator is that $C|n\rangle = |n+1\rangle$. All this has been standard material.

The quantization of fields with positive definite quadratic Hamilonian densities (this includes the electromagnetic field and presumably scalar fields too) then proceeds by defining a creation C_a and destruction $\overline{C}_a$ operator corresponding to every possible eigenmode (indexed by a) of the field (for the EM field, there are two eigenmodes, one per polarization, per allowed wavevector), such that

$$C_aC_b - C_bC_a = 0, \quad \overline{C}_a\overline{C}_b - \overline{C}_b\overline{C}_a = 0 \tag{6}$$

$$C_a\overline{C}_b - C_b\overline{C}_a = \delta_a^b \tag{7}$$

$$N_a = C_a\overline{C}_a, \quad 1 + N_a = \overline{C}_aC_a \tag{8}$$

$$H = (N_a + \frac{1}{2})\hbar\omega = (C_a\overline{C}_a + \overline{C}_aC_a)\frac{\hbar\omega}{2} \tag{9}$$

leading to the usual conclusion that each mode with occupation number n has energy $(n + \frac{1}{2})\hbar\omega$.

The situation for fermionic quantum fields, in particular, for the electron/positron field (obeying the Dirac equation) is somewhat different. Now, the creation and destruction operators instead obey

$$C_a C_b + C_b C_a = 0, \quad \overline{C}_a \overline{C}_b + \overline{C}_b \overline{C}_a = 0 \tag{10}$$

$$C_a \overline{C}_b + C_b \overline{C}_a = \delta_a^b \tag{11}$$

$$N_a = C_a \overline{C}_a, \quad 1 - N_a = \overline{C}_a C_a, \tag{12}$$

$$H = (N_a - \frac{1}{2})E = (C_a \overline{C}_a - \overline{C}_a C_a)\frac{E}{2} \tag{13}$$

where E is the relativistic energy of that mode. For a Dirac mode with momentum $\vec{p}$ we have $E = \sqrt{p^2c^2 + m^2c^4}$ where m is the rest mass of the electron; for a massless electron this would reduce to $E = \hbar\omega$. The key thing here is that since $(C_a)^2 = 0$ due to the *anti*commutation in EQ 10 (the corresponding relation for bosons is ordinary commutation), we cannot create *two* particles in the same state; there can only be either 0 or 1 particles in a mode. (This is the "Pauli exclusion principle" underlying "Fermi statistics" as opposed to "Bose statistics.")

Although everybody knows that the zero point energy for boson modes is $\hbar\omega/2$ (and more generally $E/2$), it is less well known (it is not mentioned[8]in numerous quantum field theory books) that *the zero point energy for fermion modes is* $-E/2$.

3 Connection between supersymmetry and zero cosmical constant

It is fascinating that the zeropoint energy for fermions is *negative.* This suggests that the net zero point energy of all fields, fermionic and bosonic, could exactly cancel out, explaining the fact that the vacuum weighs nothing and the $\Lambda = 0$ mystery.

Now, there is a hypothesis popular among particle physicists and string theorists, called "supersymmetry" [30]. This postulates that there is some symmetry of the laws of nature which

1. takes one-particle states to one-particle states (always mapping to a different particle with a spin differing from the original's by 1/2)
2. acts on many-particle states as though they were products of one-particle states, and
3. leaves scattering amplitudes unchanged.

(This list is quoted from the encyclopedia [41] entry by C.P.Burgess on "supersymmetry.") If so, then for every boson zero point state with energy $+E/2$ there is automatically a fermion zero point state with energy $-E/2$, and so all the zero point energies *must* cancel out!

Although I was rather proud of having thought of this... not surprisingly, it was noticed before. The first to point it out probably was one of the original supersymmetrists, Bruno Zumino in 1975 [71]. My derivation above has the advantage that it requires only very elementary knowledge, whereas Zumino's was based on evaluating an infinite number of Feynman integrals.

(In the further discussion below, I will use various numbers about particle masses, supersymmetry, and the Higgs boson, all extracted from [50].)

4 Broken supersymmetry

Before the reader becomes too excited, I warn him that the rest masses of particles and their supersymmetric partners must[9] differ (otherwise the experimental evidence would already seem sufficient to rule out supersymmetry), supposedly due to "spontaneous symmetry breaking." In that case, the $+E/2$ will *not* exactly cancel the $-E/2$.

This too had been noticed by Zumino [71] in 1975, e.g. on his p.336: "Unfortunately, when supersymmetry is broken, even spontaneously broken, the result is no longer valid." The Λ problem was thus regarded as not solved by supersymmetry, but merely altered to a new form of why something *else* which ought to be very large, happens to be very small. Weinberg in his 1989 review [68] even went so far as to write that "the effect of supersymmetry would be to convert the problem of the cosmological constant from a crisis into a disaster."

But: I am now going to make a simple observation that goes beyond Weinberg and Zumino to show that the situation is better than they had thought.

Let us make a numerical estimate of the imperfection of the cancellation. Assume we are in flat space (or a box of vacuum with periodic boundary conditions) and that the mode spectrum is cut off at the Planck energy

[8]It may be deduced from Dirac [17] but with the "infinite constant" he neglected to get his eq11.8 page 57 put back into that equation mode by mode. (His integrand in that equation consists of a sum of 4 modes, namely 2 electron spins and 2 positron spins.) The fact that suitable C and $\overline{C}$ operators can, self-consistently, exist satisfying all Dirac's relations is made apparent by

$$C = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}.$$

Incidentally, it should be remarked that one cannot get rid of zero-point energies by simply redefining the zero of one's energy scale, for several reasons: (i) one would have to move one's energy scale by a different amount for each ω. (ii) There is a *unique* finite Hamiltonian density depending only on the field and its first derivative for the Maxwell and Dirac fields, and this density leads to the given zero point energy formulae. (iii) As we've said, zero point phenomena have been observed experimentally. See also [43]. One *could* get rid of zero point energies by postulating that there are only a finite large number of modes (Dirac's treatment had assumed an infinite number) in which case a single huge constant added on at the end would do the job, but it would have to be exactly the right value, and it then would be mysterious where that constant should come from and how it "knew" to be exactly the right value to exactly cancel out all the zero point energies. Furthermore, in this scheme, this "constant" would presumably have had to be position dependent in a curved spacetime.

[9]Incidentally, the only known cases in which bosons and fermions had the same (zero) mass have been overthrown by the discovery at Kamiokande that neutrinos have mass. That perhaps grants extra interest to footnote 7.

$E_{\text{planck}} = \sqrt{\hbar c^5/G}$. Then the relative error in the cancellation will be of order (in units with $c = 1$)

$$\Big(\sum_{\substack{\text{fermion}\\ \text{modes}}} \int_{|\vec{p}|^2 < p_{\text{planck}}^2 - m_f^2} \sqrt{|\vec{p}|^2 + m^2} d^3\vec{p} - \sum_{\substack{\text{boson}\\ \text{modes}}} \int_{|\vec{p}|^2 < p_{\text{planck}}^2 - m_b^2} \sqrt{|\vec{p}|^2 + m^2} d^3\vec{p} \Big) \Big/ \int_{|\vec{p}|^2 < p_{\text{planck}}^2} |\vec{p}| d^3\vec{p} \tag{14}$$

It is possible to evaluate all the integrals in closed form. I will spare you the details; suffice it to say that the final estimate of the relative error, expanded into a series in powers of M^{-2}, where for conciseness we write $M = m_{\text{planck}}$ or m_{cutoff}, is[10]

$$0 + \frac{\sum_f m_f^2 - \sum_b m_b^2}{M^2} + \frac{4\ln 2 - 1}{8} \frac{\sum_f m_f^4 - \sum_b m_b^4}{M^4} + \frac{\sum_f m_f^4 \ln(M/m_f) - \sum_b m_b^4 \ln(M/m_b)}{M^4} + \frac{\sum_b m_b^6 - \sum_f m_f^6}{8M^6} + 5\frac{\sum_b m_b^8 - \sum_f m_f^8}{128M^8} + \cdots \tag{15}$$

It is usually suspected that the (so far undiscovered) superpartner particles have rest energies $mc^2 \approx 1$TeV. In the limit $M \to \infty$ we would get perfect cancellation, but M is only 16 orders of magnitude larger than 1TeV so we only get, from the second term of EQ 15, that our relative cancellation error is of order $\lesssim 10^{-32}$.

Although cancellation of 32 significant figures is impressive, it is not enough by itself to explain the observed cancellation, which is 120 significant figures assuming a Planck cutoff[11].

If the sum of the squares of the masses of the fundamental fermions, happened to be exactly equal to the sums of the squares of the masses of the fundamental bosons (where in the sums, different spin and charge states are counted multiply), i.e. if

$$\sum_{\text{bosons}} m_b^2 = \sum_{\text{fermions}} m_f^2, \tag{16}$$

then the second term in the error estimation series EQ 15 also would vanish, leaving the next terms, i.e. now yielding cancellation of the *64* most significant decimal places.

Is there any hope that EQ 16 could be true? There are two answers to that: theoretical and experimental.

[10] The same series (EQ 15) still would hold to this order of accuracy, if the upper limits in the integrals in EQ 14 were all replaced by p_{planck}^2 exactly so that the same number of modes are integrated over for each particle type – *except* that the coefficients of the M^{-2} and M^{-6} terms in EQ 15 then need to be negated. Our conclusions would not be affected.

[11] One could defeat this objection and still obtain precise cancellation by postulating (also a popular speculation among theorists) that the "unrenormalized" or "bare" rest masses of all particles are each *zero*, and that bare masses are the correct ones to use when computing the weight of the vacuum. But I do not agree with this because it seems peculiar to me that for real particles one should use the true mass but for virtual particles one should use the bare mass.

5 The theoretical answer

The theoretical answer was found by Ferrara et al. [22] in 1979. It is that, in highly general *broken* supersymmetry theories [53], *an identity equivalent to EQ 16 holds.* I am pointing out the application of this identity to the cosmical constant problem for the first time. (At least, this had remained unnoticed by [68] and other references on that problem[12].) I regard this as highly suggestive that we are on the right track; surely it is not happening for nothing.

This identity was only claimed [22] to hold "at tree level," i.e. to first order in an asymptotic series expansion in powers of the (unknown small) supersymmetric coupling constant. In broken supersymmetry it holds [53] to "first order in the [reciprocal of the unknown large] supersymmetry breaking." This would seem to indicate that the remaining error in (EQ 15) was not of order $(m/M)^4$, but rather of order $(m/M)^2(m/m_{\text{SUSY}})$, or perhaps $(m/M)^2(m/m_{\text{SUSY}})^2$ (since we are considering a formula *even* in all the particle masses) or $\alpha_{\text{SUSY}}(m/M)^2$ or $(\alpha_{\text{SUSY}})^2(m/M)^2$, where m ($m \approx 1$TeV?) is the largest particle mass, M is the Planck mass, m_{SUSY} is the "supersymmetry breaking mass scale," and α_{SUSY} is the dimensionless supersymmetric coupling constant (perhaps about 0.03, cf. p.787 of [52]).

In recent papers $m_{\text{SUSY}}c^2$ has been claimed to be anywhere from about 100GeV up to 10^{16}GeV. Specifically, I have seen the following quoted in recent papers: 182.5, 10^3, 10^{11}, 10^{15}, and 10^{16} GeV, with all but the first value being ascribed to "common convention!"

Fortunately, it will suffice for our purposes in this paper (except in §12.3) merely to assume that the Ferrara identity EQ 16, and hence our energy cancellation in EQ 15, are valid down to relative error substantially smaller (a factor of 10-100 smaller should be quite safe; probably even a factor of 2 would be sufficient) than the typical summand in EQ 16.

[12] E.g. in Sahni's (2000) review [56] (page 567): "There was some hope during the 1970s that supersymmetric theories might reduce the value of Λ_{vac} since fermions and bosons contribute with opposite signs [Zumino1975]. However since supersymmetry is broken on scales $< 10^3$GeV it is unlikely one can appeal to this mechanism to generate the small value of Λ which is observed today." Similarly in the excellent 1995 quantum field theory textbook [52] (page 797): "In a grand unified theory with broken supersymmetry, those momentum scales that are much larger than the mass differences of supersymmetry partners give no contribution to the vacuum energy. Thus the natural size of Λ in these theories is $\approx (100GeV)^4$. This reduces the cosmological constant problem to a discrepancy of 50 orders of magnitude – but this is not nearly enough." Among the 111 papers citing [22], the ones coming the closest to the present ideas were [35] (see p.561) and the original paper [22] itself. But neither of these papers shows awareness of the importance of "spectral roughness," neither one is centered on our topic, and I also think that although both show appreciation of the fact that EQ 16 has implications for cosmical constant reduction, they come to that appreciation in different, weaker, and more complicated ways than I do. For example, my EQ 15 is based on an *exact* integration of EQ 14 and then expanding about an infinite frequency cutoff, and is not the usual kind of *perturbative* result one gets from evaluating Feynman diagrams.

6 The (As yet incomplete) experimental answer

Unfortunately, there is no *direct* evidence for supersymmetry, i.e. not a single supersymmetric partner has been found in accelerator searches (present lower limits on their masses range from 23-260GeV, depending on the particle) and known fermions presently outnumber known bosons by over a 3:1 ratio.

But there is one kind of convincing *indirect* evidence (dark matter) that particles unknown to the standard model must exist, having properties apparently compatible with being superpartners; there is a second line of evidence (ultra high energy cosmic rays) which is less convincing, and finally there is a third kind of evidence (muon magnetic moment) which is presently unconvincing. These will now be reviewed.

6.1 Dark matter

Supersymmetry predicts that there must be at least one (as yet undiscovered) stable massive particle free of every kind of charge and hence extremely non-interacting. Such particles are called "WIMPs." They are not part of the standard model. They might only be detectable gravitationally.

Vast amounts of "dark matter" have been detected through its gravitational effects. This matter seems to be distributed in roughly spherical "blobs" centered at the locations of galaxies, and its distribution (inferred either by star motion measurements, or by gravitational lensing; both agree[13] is entirely different from the distribution of the visible matter in those galaxies (which is lens shaped, with smaller diameter than the blob). The local matter density in our own galaxy in the neighborhood of the solar system has also been inferred from recent accurate parallax measurements of the proper motions of nearby stars by the Hipparcos astrometry satellite [51]. Unless our solar system's local neighborhood is highly atypical of this radius out in the galactic disk, this method too agrees that the matter in the disk itself cannot be responsible for most of the gravity holding our galaxy together.

This dark matter presumably is different than the luminous visible matter in galaxies, otherwise why would it not form stars and why would it not adopt a len shaped distribution also? Thus, the dark matter (which seems to constitute 90-99% of the mass in the universe!) presumably must be made of WIMPs [37][58][64].

6.2 Ultra-High energy cosmic rays

Numerous "ultra high energy" ($> 8 \times 10^{19}$ev$+1\sigma$ and $> 5 \times 10^{19}$ev$+2\sigma$) cosmic rays (UHECRs) have been detected, but, according to G.Farrar et al. [21] so far only five have had their directions resolved within < 10 square degrees worth of solid angle on the celestial sphere. All five of them pointed back to within 183.4 arcminutes of a "compact radio-loud quasar" from the previously compiled (by NASA and the Max Planck Institute) list of about 600 such objects. The chance that this would have happened by luck (if the true sources of the rays were random on the sphere, as opposed to the hypothesis the quasars were the sources) is $[(183.4/60/360)^2\pi^2 600]^5$, i.e. < 0.015. A more careful likelihood analysis involving the 5 precise angular separations and (assumed Gaussian) error distributions for the cosmic ray directions, is claimed [21] to yield 0.005. These 5 quasars have redshifts ranging from $z = .29$ to $z = 2.18$, indicating the closest is $>$ 1000megaparsec away. But it is believed that the universe is opaque to electrons (and positrons), protons, atomic nuclei, or photons with this large an energy (due to, respectively, inverse Compton scattering, resonance exciting and pion production, photodissociation, and e^+e^- pair production interactions with microwave and radio background photons) with a mean absorption length of $<$ 12MPc [25][19] (also, nuclei, protons and especially electrons might have been bent by magnetic fields so that they would not have come from these directions anyway). Therefore, we conclude (with 99% confidence) that these 5 cosmic rays must have been some particle other than electrons, protons, nuclei, or photons. The detection signatures and known cross sections and collision depths in the atmosphere are believed to rule out the possibility they were neutrinos. They could not have been neutrons because neutrons have a mean lifetime before decaying of only 890 seconds – not long enough to have survived their journey to Earth even taking the enormous Lorentz time dilation factors into account. (Particle lifetimes $\geq$ 1week seem required.) Comparing air shower simulations with the detection signatures suggests the rays were hadronic and with rest masses $\leq 50\text{GeV}/c^2$.

All this tells us that these cosmic rays must have been some new, stable (for $\geq$ 1week anyway) electrically neutral (to prevent bending by magnetic fields en route, and to reduce interaction with cosmic background photons) particle of mass $> 2\text{GeV}/c^2$ (to prevent excitation of "resonances" by cosmic background photons en route from sapping its energy too severely) susceptible to the strong force. The standard model does not include any such particle. But Farrar [21] has argued that it could have been a composite particle called the S^0 predicted by supersymmetry[14].

In my opinion, Farrar et al.'s argument was both insufficiently careful and overstated. Despite this, the argument (if redone more carefully with more data) probably still would have a good chance of being valid.

1. Essentially every claim made by Farrar et al.

[13] Also, simulations of time evolution of galaxies indicate stable spiral shapes would not have occurred in the absence of a dark matter halo; also galactic clusters would be unable to hold themselves together without dark matter's gravity.

[14] Another "hybrid" possibility suggested by T.J.Weiler was that it was a neutrino which, somewhere near our galaxy, hit some dark matter and ejected hadrons, which were the detected cosmic rays; but this hypothesis also needs to involve new, supersymmetric particles.

about propagation distance of UHECRs of various possible kinds (photons, protons, etc.) was *wrong*, obsolete, deceptive, and/or inaccurate. For example, [14]: "even photons of such high energies [apparently referring to the highest energy UHECR ever seen, 3.2×10^{20}eV] have a mean free path of less than 10 Mpc due to scattering" is misleading since (a) what matters is the energy decrease length, not the mean free path between scatterings, and that is 16 MPc at 10^{19}eV, 13 MPc at 10^{20}eV, and 100 MPc at 10^{21}eV according to figure 11 in [8] (b) this sentence seems to be intended to give the impression that photons have the largest absorption lengths, hence if UHE photons cannot propagate far, neither can anything else. But actually, both protons and iron-56 nuclei at 10^{20}eV have larger energy-loss lengths than 10^{20}eV photons.

Upon correcting these claims, in every single case, the propagation distance increases, making it less necessary to invoke new particles beyond the standard model to allow the sources to have been the 5 quasars. In particular, the two corrections which matter the most for us are: (a) previous calculations of distance limits for ultrahigh energy protons due to photopion production had been based on wrong photopion cross section versus energy curves and/or a wrong spectrum for the photon background; and (b) previous calculations of distance limits for ultrahigh energy nuclei due to photodisintegration had been based on wrong (too large) notions of the infrared background. For all this see the excellent (and up to date) survey paper (which has many nice attenuation length versus cosmic-ray energy plots in it) [8]; see also [42][61].

Even these *corrected* calculations of energy-loss lengths may be good enough for Farrar's purposes – but they may not be. Thus ([8] fig. 9), $1/e$ energy attenuation lengths for nuclei and protons from photopion production are: 20000 Mpc at 10^{19}eV, 70 Mpc at 10^{20}eV, and 19 Mpc at 10^{21}eV, and are 200 Mpc at Farrar's 1-sigma energy threshhold 8×10^{19}eV. According to figure 3 of [61], iron-56 nuclei from 1000 Mpc away could plausibly make it to Earth without being disintegrated if they were below 4.5×10^{19}eV (5×10^{19}eV was Farrar's 2-sigma energy cutoff). Since Farrar is considering some of the highest energy cosmic rays ever detected, it is not unreasonable to assume they all were "tail events" featuring anomalously large energy-measurement errors and/or which survived for an anomalously large distance. Even if the probability were only 10^{-6} that each of these anomalies happened, that would not matter much, logically, since we are only interested in *conditional* probabilities conditioned on the fact that these rays were observed. So it is necessary to do a careful statistical analysis with these notions in mind, and I claim Farrar et al. did not.

2. I pointed out to Farrar at least 8 other UHECR events besides her 5. How do they affect her analysis? She never said. In situations such as this, it is essential to incorporate as much data as possible.

3. Farrar gave no consideration to the possibility that the "UHECRs" were in fact small (10^{-14}gram) grains of antimatter.

4. It may still be necessary to invoke S^0's (or some neutral particle anyhow) to avoid bending of the cosmic rays off course by magnetic fields (since the quasar data indicate they were not bent off course). Thus, according to the rough approximate formula for bending angle Farrar and Bierman [21] quoted as their EQ 1, if their "scale length" λ exceeds .2 megaparsec (c.f.: radius of milky way = 8 megaparsec) and the RMS magnetic field exceeds 1 nanoGauss, then UHE protons and nuclei would have been bent off course by intergalactic magnetic fields. Although this approach sounds promising, its crux would need to involve understanding the magnitude and statistical behavior, especially correlation length, of extragalactic magnetic fields – a topic about which we know very little. Estimates of intergalactic magnetic fields have ranged from below 10^{-13} Gauss to as much as 5×10^{-6} Gauss, and speculations have been made that this field is made of random domains rather like a ferromagnet, or instead that has a "sheet and filament" structure. In short, not enough seems known to draw a confident conclusion.

6.3 Muon moment

The recent "E821" experiment at Brookhaven [12] measuring the magnetic moment of the muon extremely precisely has supposedly found a $> 2.6\sigma$ discrepancy (which, with Gaussian statistics, would correspond to $> 99.5\%$ confidence) between the experimental g-factor value $2.0023318400(28_{\text{syst}})(12_{\text{stat}})$ and the theoretical prediction 2.0023318320(13) of the "standard model." This has been claimed to be consistent with supersymmetry but inconsistent with the standard model alone. Because of the high level of trust I am asked to place in these experimenters and theorists and their error analyses, I presently do not find this to be convincing evidence for the failure of the standard model. However, about 4 times more data, which should reduce the experimental *statistical* error by a factor of 2, as well as measurements on the anti-muon, and a re-done theoretical evaluation, have been promised in one more year.

7 Experimental answer within the standard model alone

If we disregard all speculations about supersymmetry, and merely inquire about the masses of "standard model" particles, then the situation is as follows.

Thanks to the top quark mass $174.3 \pm 5.1\text{GeV}/c^2$ (present in 12 versions: spin $\pm 1/2$ and charge $\pm 1/3$ and 3 colors), known fermions have larger $\sum m^2$ than known bosons, although if the top quark were disregarded, the bosons (thanks to the W and Z bosons with masses 80.419 and $91.188\text{GeV}/c^2$ respectively, and present in 6 and 3 versions respectively) would be winning. The only still-undiscovered particle predicted by the "standard model," the Higgs boson, thus would indeed lead to a tie if it had a mass $m_{\text{needed}} \approx 549\text{GeV}/c^2$. (Although then, standard model fermion modes would still heavily outnumber standard model boson modes; only supersymmetric extensions of the minimal standard model would allow these counts also to be equal.) This lies outside the partly theoretical and partly experimental 95%-confidence bounds on the Higgs mass: $108 \le m \le 215\text{GeV}$ (more conservative bounds, based on more-direct experimental evidence, are $60 \le m \le 800\text{GeV}$).

8 Interactions between fields

Another caveat in all this is that in our treatment of fermion and boson zero point energies and creation/destruction operators, we only considered "non-interacting" quantum fields. Would interactions alter the picture? Consider the interaction between an electron/positron and an EM zeropoint vacuum field mode. There should be an exact cancellation of the positron-EM and the electron-EM interaction energies in each Fourier mode. Similarly in any interactions between fermion and boson vacuum field modes, the anti-fermion contributions should exactly cancel the fermion contributions in each mode. Every boson in the standard model has an equal and oppositely charged version so that a similar remark apply to the interactions between them, and one might at least hope that this would also be true of any further non-standard bosons that might be discovered in the future. (And there are no fermion-fermion interactions in the standard model – these being always "mediated" by bosons.) So apparently interactions have zero net effect at each wavevector and hence are a non-problem.

9 Further cancellations?

Supersymmetry automatically gives us about 32 decimal places of cancellation just due to the fact that the counts of Fermion and Boson modes, up to any given wavenumber, are equal. We've now seen that the cancellation of the second term in the asymptotic series (EQ 15) now really produces 64 decimals of cancellation.

But this is still not enough, since we need 120. Might there be some additional identity causing the M^{-4} terms of EQ 15 also to vanish, so that we get 96 decimals, and then perhaps also $\sum_{\text{bosons}} m_b^6 = \sum_{\text{fermions}} m_f^6$ so that the *next* term would vanish, giving us 128 decimals, and so on? This seems unlikely[15] – but we will see that it is actually not necessary, and even if it did happen it would have no useful effect. That is because the whole asymptotic development (EQ 15) really is not valid beyond its second term anyway, because the true error, after the first and second terms are removed, does *not* behave like the third term in that series, but rather is (a) far larger and (b) "rough." I call this phenomenon "spectral roughness[16]."

10 Spectral roughness

Really our *integrals* in EQ 14 are only approximately correct, and what would be precisely correct (if the universe really were a box with periodic boundary conditions) are the spectral *sums* these integrals are approximating. Specifically, each "mode" has a zeropoint energy $\hbar\omega/2$, where the angular mode frequencies ω are the eigenvalues of appropriate wave operators, and the total energy is the *sum* of these.

In the case of the Laplacian in a box of sidelength L with periodic boundary conditions we can make this very explicit. That is because the spectrum is then precisely known, since the eigenfunctions are just Fourier modes $\exp(2\pi i\vec{k}\cdot\vec{x}/L)$ with 3D integer wavevectors $\vec{k}$ and corresponding eigenfrequencies[17] $\omega = 2\pi c|\vec{k}|/L$.

Thus if we want to sum the mode energies up to some cutoff value U (and for simplicity and concreteness let us here just consider the Laplacian operator rather than, e.g., the Dirac operator) we want to compute an energy-sum of the form

$$E(U) = \frac{hc}{L} \sum_{|\vec{k}|\le U} |\vec{k}|, \tag{17}$$

i.e., just the sum of the lengths of all the lattice 3-vectors in the ball of radius U. Asymptotically for large U, (by approximating it by $\frac{hc}{L}\int_0^U r\cdot 4\pi r^2 dr$) this sum obviously will behave like $\frac{\pi hc}{L}U^4$. This is a pleasantly smooth function of U.

To simplify the below discussion, let us use units in which $c = L = h = 1$. Then our integral approximation just becomes $E(U) \approx \pi U^4$.

If we now look at the situation more precisely because we are interested in higher-order terms in the asymptotic expansion, we discover "roughness;" obviously the sum really yields not πU^4 but instead a "staircase function." A plot of $[E(U) - \pi U^4]/U^2$ for integer U^2 is shown in figure 1[18].

[15] Comment by the referee: In [16] it is argued that the sum-rule (EQ 16) cannot hold exactly because if it did, some superpartners would be lighter than ordinary particles and hence should already have been observed. In a recent model of "gauge-mediated supersymmetry breaking," [29] one can explicitly check that the resulting correction to (EQ 16) should be of order TeV^2. There will be violations at "loop level" (i.e. the next level of approximation after the present "tree level" at which EQ 16 was shown valid [22]).

[16] The whole notion of "roughness" in the asymptotic behavior of spectral sums, preventing their asymptotic treatment beyond some order of accuracy, will be very important to, and will be discussed elsewhere in, this series of papers. It may also arise elsewhere in physics. Here I'll just discuss the case at hand.

It is important to realize that the referee's comment in the previous footnote does *not* refute the present paper. Due to my arguments about "spectral roughness" and "feedback," it is only necessary for me that (EQ 16) hold approximately. Violations at loop-order are perfectly fine with me.

[17] The angular eigenfrequency of the wave equation $\ddot{F} = \nabla^2 F$ is ω if the Laplacian's eigenvalue is $-\omega^2$.

[18] If you want to make your own such plot, you will save a vast amount of computation by noting that the number $N(m)$ of $\mathbf{Z}^d$

Figure 1: A plot of $[E(U) - \pi U^4]/U^2$ for integer U^2, $1 \le U^2 \le 10^6$. (The horizontal axis is $\log_{10} U$.) Here $E(U) = \sum_{|\vec{k}| \le U} |\vec{k}|$ where the sum is over integer 3D lattice points $\vec{k} \in \mathbf{Z}^3$. Consecutive points are joined by line segments as an aid to the eye (really they should be joined by certain piecewise polynomial curves, but that distinction is mostly invisible). This plot demonstrates "spectral roughness" in the sum $E(U)$ of wave-mode energies below cutoff-frequency U in the cube with periodic boundary conditions.

As you can see, it fluctuates in a random-looking manner, apparently changing sign infinitely often and making excursions away from 0 in both directions by unboundedly large amplitudes (which appear to grow with U proportionally to $\log U$). Furthermore, the *density* of large, sign-changing fluctuations on the U axis seems to tend to ∞ (roughly proportionally to U). As we'll see in §11, statements only slightly weaker than these empirical observations can in fact be proven by taking advantage of years of study by many mathematicians [27][66][67][38][23][20][28] , going back to C.F.Gauss, on the closely related "lattice point counting error problem."

11 Some known results on 3D lattice point counting error problems

Let $N(R)$ be the number of 3D points with integer coordinates lying in the ball (centered at $\vec{0}$) of radius R, let $V(R) = 4\pi R^3/3$ be the volume of that ball, and let $\Delta(R) = N(R) - V(R)$. The "lattice point counting error problem" is to describe the behavior of $\Delta(R)$.

Instead of a ball, one may also consider an arbitrary particular $\vec{0}$-centered *ellipsoid* of volume $4\pi/3$, in place of the unit ball, and consider scaling it by R. In another variant, we consider an arbitrary particular *smooth convex body*[19] containing $\vec{0}$ in its interior, scaled by R.

11.1 Upper bounds

Obviously $|\Delta(R)| < 4\pi\sqrt{3}R^2$ for the ball problem, and indeed $|\Delta(R)| < \sqrt{3}S(R)$ for an arbitrary smooth convex body, where $S(R)$ is the surface area of the R-scaled body.

Hlawka [33] in 1950 showed, more strongly, that $|\Delta(R)| = O(R^{3/2})$ for an arbitrary smooth convex body, and that here the "3/2" cannot be replaced by 1. The exponent 3/2 in Hlawka's upper bound recently was reduced [39] to $25/17+\epsilon$ (throughout this discussion, ϵ will denote an arbitrarily small positive real). An especially large amount of analytic attention has been devoted to the ball case, for which the latest and smallest exponent upper bound [32] is $21/16 + \epsilon$. Probably the true answer is $1 + \epsilon$ for all problem variants.

11.2 Statements about the mean square

Jarnik [36] in 1940 showed, in the case of an arbitrary particular ellipsoid, and for a certain explicitly computable constant K, that

$$\int_1^T \Delta(R)^2 R \mathrm{d}R = KT^4 \log T - O(T^4 \log^{1/2} T). \qquad (18)$$

This is consistent with the heuristic notion that $\Delta(R)$ asymptotically is "random with variance $4KR^2 \log R$." Bleher and Dyson [10] indeed conjectured it asymptotically had a Gaussian probability density with this variance.

11.3 Lower bounds

Again, the strongest theorems presently known apply only to the ball. A superb result by Tsang [65] shows that

$$\sup_{X<R<X+X^\epsilon} \pm\frac{\Delta(R)}{R} > \sqrt{\epsilon \log X} \cdot \mathrm{const.} \qquad (19)$$

for any $\epsilon > 0$, any choice of the $\pm$ sign, and all sufficiently large X. One immediate consequence is that $\Delta(R)$ must change sign in every interval $(X, X + X^\epsilon]$, for all sufficiently large X. Tsang was able to show, more strongly, that some large positive constant W existed, such that $\Delta(R)$ must change sign in any interval $(X, X + W]$ for all sufficiently large X.

Probably Tsang's arguments and results could be generalized to hold for an arbitrary ellipsoid, but that has not been done[20]. The best results currently known in the ellipsoid case are [3]

$$\limsup_{R\to\infty} \frac{\Delta(R)}{R \log\log R} > 0 \qquad (20)$$

lattice points at distance $\sqrt{m}$ from the origin in $\mathbf{R}^d$ (for us, $d = 3$) is the coefficient of q^m in $\theta_3(q)^d$ where $\theta_3(q) = 1 + 2\sum_{j\ge1} q^{j^2}$.

[19] For us, a "smooth" convex body means one with positive Gaussian curvature everywhere on its surface, a uniquely defined surface normal everywhere, and such that the map between its unit surface normal vector and the point on the surface, is 1-to-1 and infinitely differentiable.

[20] Yuk-Kam Lau and Tsang will soon publish a paper in Crelle's journal extending Tsang's result to any ellipsoid defined by a quadratic form with *rational* coefficients.

and the following holds for an arbitrary smooth convex body [49]

$$\liminf_{R\to\infty} \frac{\Delta(R)}{R\log^{1/3} R} < 0. \tag{21}$$

11.4 *Counting lattice vectors, versus summing their lengths*

All of the above results about *counting* lattice vectors in regions, unfortunately do not apply to the quantity we really care about, namely the *sum* of their lengths. Fortunately, though, *Tsang's* bound EQ 19, due to the fact that it provides *local* information, can easily be used to produce such a result, namely: for our cube with periodic boundary conditions model universe, the function $[E(U) - \pi U^4]/U^2$ plotted in figure 1 indeed does exhibit fluctuations with unbounded amplitude (in both directions) growing with U at least with order $\log^{1/2} U$. (This is slightly weaker than $\log U$, which appears, empirically, to be the truth.)

Furthermore, at least one peak-to-peak swing, of amplitude of order $\pm\log^{1/2} U$ (or larger) to amplitude of this order but the opposite sign, occurs on every set of the form $X < U < X + X^\epsilon$, for each $\epsilon > 0$ and all sufficiently large X. (Empirically[21], such swings usually occur on intervals of width only $4/U$.)

12 Spectral roughness swamps terms in our energy series

12.1

Using the Tsang-consequence roughness lower bound from §11.4 with $\epsilon = 0.01$, this means that perturbing the sidelength L of our cubical model universe by $\pm(L/\ell_{\text{planck}})^{0.01}\ell_{\text{planck}}$, would always suffice (for all sufficiently large L) to cause a large sign-changing swing. If $L = 10^{10}$ light years, the required length perturbation would be only $4.05\ell_{\text{planck}} \approx 7 \times 10^{35}$ meter. The amplitude of the energy change caused by such a swing, would actually be of slightly larger order (by a factor going to infinity with the energy cutoff U, but only logarithmically) than the second term of the asymptotic series EQ 15. Thus even this term, at least in our cubical universe model, is not really valid, since it is swamped by the roughness. Since in practice $U < \infty$, indeed we are taking $U \approx L/\ell_{\text{planck}} \approx 10^{60}$, and since the swamping factor is growing only like $\log^{1/2} U$ or $\log U$, it is not completely clear that this term would really be swamped. Thus it is a considerable comfort that by EQ 16 it is, in fact, zero. The fact that the *next* (third) term is swamped, *is* clear.

[21]This empirical observation suggests that our assumption of a *finite* universe was in no way essential. As the universe-size grows toward ∞ and the wavelength cutoff ℓ_{planck} shrinks toward 0 (or either one), all my arguments about spectral roughness and asymptotics seem merely to become more true. See point 2 in §14.2 re this.

12.2 *Does this depend on having a* cubical *model universe?*

All this reasoning could be attacked by arguing that Tsang's results, leading to our rigorous roughness bounds, are *specialized* to the case of the ball-lattice counting problem and hence to the *precisely cubical* model universe. The cube is a very symmetrical object, and so is the ball, and this (it might be worried) could lead to exceptionally large roughness, since the "staircase" will contain jumps of exceptionally large heights, due to the presence of many integer vectors having the same length[22]. To make all this more realistic (the attack would continue) really we should study, *not* a model universe that is a perfect cube with periodic boundary conditions, but instead some more generic and less symmetrical manifold. (Unfortunately: less symmetric manifolds are harder to study.)

As a first step in the direction of genericity one instead could consider a *parallelipiped* with periodic boundary conditions as our model universe, with the corner coordinates of the parallelipiped chosen to be generic algebraically independent irrationals. The mode energy sum would then correspond to the problem of summing the norms of the lattice points lying inside a (generic $\vec{0}$-centered) *ellipsoid*, using a norm based on that ellipsoid's defining quadratic rather than the usual L_2 distance. It seems probable from EQ 20 and 21 that here again the spectral roughness would, asymptotically, *still* swamp even the second term of EQ 15 (although by even slower growing unboundedly large factors). Although my attackers are correct that the cube leads to atypically large jump heights in the staircase (while generically no jump should be larger than a constant, since generically there should not be any coincident eigenfrequencies – this genericity property is massively violated by the cube), that does not matter since other effects besides these jump heights contribute enough amplitude to the roughness, so that our conclusions remain unaffected.

Still more generally, one could consider highly arbitrary smooth compact 3-manifolds as universe models. And indeed, unfortunately for me, there is reason to suspect that such manifolds would exhibit *less rough* spectra than paralleliped flat-tori; instead of having roughness of amplitude of order $\approx U^2$ where U is the cutoff frequency, the amplitudes would be only of order U^1 (both up to logarithmic factors). This would no longer be large enough to swamp the second term (were it nonzero) of EQ 15, but it would still be enough to swamp the third (order U^1) term. Indeed its amplitude is roughly midway, on a logarithmic scale, between that of the second and third terms. The "reason to suspect" this is: common beliefs about the nature of "quantum chaos."

[22]Indeed, the 3-sphere surface of the 4D ball also has precisely calculable eigenmodes (spherical harmonics) and it appears to be the unique 3-manifold with maximum "spectral roughness" – providing an example of a 3-manifold for which my arguments work *better* than for the cube.

12.3 Quantum chaos

Some physicists believe that spectral roughness generically exhibits statistical "universality" properties – this has been called "quantum chaos." Specifically, let the "spectral rigidity" of a set of energy levels be the mean square deviation of the spectral staircase (but with height-1 steps at each energy level, unlike my staircase) versus the straight line that best fits it over a range of N consecutive levels. This is a measure of how much these levels "fluctuate." Berry [6] argued that generic physical systems with chaotic classical behavior (i.e.: most systems; also this surely should include the geodesic flow in the entire universe!) would exhibit quantum energy levels with spectral rigidity asymptotic to $\ln(N/N_0)$, the same behavior as for the eigenvalues of matrices from the "Gaussian orthogonal ensemble" (GOE). Meanwhile classically non-chaotic systems would have spectral rigidity asymptotic to $N/15$, i.e. far larger.

Two large numerical studies were done in the mid-1980s to test Berry's belief by contrasting a simple non-chaotic and an analogous chaotic system. The non-chaotic system was simply a free quantum particle in a generic rectangle [13] with reflecting walls. The chaotic system was the same, except that an excluded circular reflecting wall was placed inside the rectangle [11]; the curvature of this wall caused the geodesic flow to become chaotic.

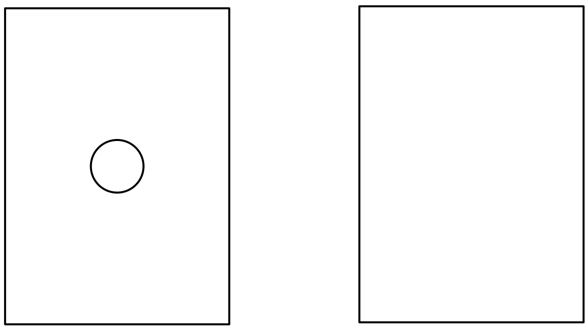

Figure 2: (left) The spectrum of a annular drum with rectangular outer boundary and circular inner boundary was studied numerically by Casati et al. [13]. It exhibits "quantum chaos" and has spectral fluctuations closely matching GOE statistics. (right) The spectrum of a generic rectangular drum was studied numerically by Bohigas et al. [11]. Classically it is non-chaotic, and its spectrum exhibits much larger fluctuations than the drum on the left.

In both cases, the first $\approx 10^5$ eigenvalues of the Laplacian were computed to high accuracy. The results were: the chaotic system indeed exhibited level spacing statistics and spectral rigidity behavior perfectly fitting the GOE predictions. Meanwhile, the non-chaotic system exhibited something similar to Poisson statistics, but significantly differing from them at both small and large N. Based on these studies (and conveniently ignoring the non-Poisson statistics as "anomalous") physicists again proclaimed conjectures of "universality."

As far as I know, none of these universality conjectures have been proven. But if we believe them nevertheless, we should expect that, in fact, generic curved manifolds will exhibit spectra with quantum chaos and thus with level-counting fluctuations a factor of approximately $N/\log N$ smaller in mean square (over stretches of N consecutive eigenvalues) than for parallelipiped flat-tori. For us, N corresponds to U^2 (since that is the threshhold at which the Poisson statistics end and beyond which the spectral rigidity becomes constant [11]), so the decrease factor for *root*-mean-square fluctuations in mode counts would be of order $U^1/\sqrt{\log U}$, presumably causing the fluctuations for energy sums also to decrease by approximately the same factor, leading to roughness of order $U^1 \log U$, rather than the empirical amplitude for a cube, $U^2 \log U$.

If this scenario of quantum chaos is believed to hold, then the arguments we will make in §13 will be placed under some strain, since to make them valid it would now be necessary to assume the relative error in EQ 16 is of order $\le m/M$, as opposed merely to assuming it is $\le .1$. (Even this assumption could be reasonable, though, if, e.g., supersymmetry energies live approximately midway, on a log scale, between elementary particle masses and the Planck mass.)

12.4 Weyl formula

The fact that the nth angular eigenfrequency ω_n of the wave equation $\ddot{F} = \nabla^2 F$ on *any* Riemannian 3-manifold of volume V obeys $\omega_n \sim 2\pi(3n/[4\pi V])^{1/3}$ was shown by H.Weyl in 1911 [54]. Consequently $\sum_{\omega_n<U} \omega_n \sim U^4 V/(8\pi^2)$. We are interested in the fluctuation behavior of the *error* in this Weyl-like formula, which is a topic which has unfortunately been less investigated, although see [26][34].

12.5 A different model for generic spectra

One approximate model of the spectrum of a manifold which has a slightly *distorted* cube-based flat-torus metric would be: not the lengths of the integer lattice vectors inside a ball implementing the ultraviolet cutoff, but rather, the lengths of those vectors each independently *randomly perturbed*, by, say, a unit-width 3D Gaussian. It is easy to see that this would leave our original spectral roughness claims from §12.1 unaltered.

13 Possible exploitation of spectral roughness by feedback mechanism to make $\Lambda \to 0$

To summarize our realizations from §10, 11, 12.1, 12.5 about spectral roughness: any attempt, as in EQ 14, to asymptotically approximate these spectral sums by a series of smooth power-law terms, *cannot succeed* beyond the second term in the asymptotic series. If you tried, then the "third" term would simply be swamped by the "noise" (spectral roughness). (Indeed even the second term would be swamped, although if so only by a slowly-growing factor.)

Indeed, even if the $E(U)$ curve were "smoothed" by averaging it over a moving window, centered at U, and of width W, then still the roughness would only be decreased by a factor of order W, which still would not be good enough – even if W were of order U (maximally large), any asymptotic series term of order U^1 or smaller *still* would be swamped in the "noise."

So as a consequence of this spectral roughness, our "cancellation error," i.e. the effective cosmical constant Λ, will be a "noisy" (pseudorandom, resembling 1D Brownian motion) function of the quantity we are varying. This quantity could be either: the box side-length of our model universe, the cutoff frequency, the Planck mass, Newton's gravitational constant G, the speed of light c, or $\hbar$ – all will have the same effect. Λ will have small absolute value everywhere, and will cross zero tremendously often.

Thus, at least in our present simplified model of the universe as a flat-space box of vacuum with periodic boundary conditions, it ought to be entirely possible for the size of the universe to be chosen (by means of a tiny – asymptotically tending to zero times the Planck length, if our empirical observations about zero crossing density are correct – perturbation to whatever value you initially had in mind) to move to a particularly favorable point in the plot of the "noise" (which is actually a deterministic spectral function that looks noisy, of course) to make the cancellation actually be *perfect.*

In any case, the cosmical constant (if there is supersymmetry) clearly *is* zero to the limit of validity of the spectral asymptotic approximations that this whole paper is based on.

Indeed, the adjustment of the "size (box side-length) of the universe" required (in our cubical-box model universe) to ensure perfect cancellation would be of order $\pm(\ell_{\text{planck}})^2/\ell_{\text{universe}} \approx 10^{-95}$meter, if our empirical observation that the zero-crossing frequency in our plot rises proportionally to U, is believed. In any case it is not of order much larger than ℓ_{planck} and that is presumably also true in generic universe models and does not depend on having a precisely cubical box model.

This still would leave us with the puzzle of why the universe box-size just happened to have such a fortuitous precise value so that $\Lambda = 0$ exactly. Spectral roughness and the cancellation of the first two asymptotic series terms had assured such values densely *exist* – but why are they *attained*?

I now propose that there is some sort of feedback-control process that made $\Lambda \approx 0$ and is keeping it there. E.g., if the universe's size were too large (or equivalently Newton's G were too small, or m_{planck}, which governs the ultraviolet cutoff, too large) then our universe model would find itself with too much "mass" inside it due to a few "extra" non-canceled high-frequency zeropoint modes. This extra mass would have a very significant gravitational effect – about 10^{56} times the effect of all the conventional mass in the universe! – which would cause our universe to quickly shrink a tiny amount, causing these extra modes (now at slightly higher frequency) now to be cut off. Similarly a deficit of modes would cause the universe to expand, causing new modes to appear below the cutoff frequency. In some such way m_{planck} and/or the universe size should effectively "self-seek" suitable values at which $\Lambda = 0$ by means of a "feedback control loop."

There might in fact be both "unstable and stable fixed points." (Specifically, the zero crossings with downward slope are one, and those with upward slope are the other.) But the feedback should of course quickly settle into one of the stable ones (which certainly should be dense everywhere, since 50% of something dense is dense too), each of which necessarily will have $\Lambda \approx 0$. Such a feedback mechanism would apply in essentially any universe, not merely our present flat-space box-of-vacuum model.

14 What would this feedback be like?

It is mysterious to me (and probably will remain mysterious until quantum gravity becomes understood) precisely how this feedback process is going to operate. Factors helping to make it mysterious include:

1. Our universe is a Lorentzian rather than Riemannian manifold (thus there would presumably be speed-of-light time delays)
2. Also the fedback shrinkage and expansion should operate locally as well as globally, e.g. any *part* of the universe with excess vacuum mode energies (whatever that means) would presumably shrink, etc.
3. How can a gravitational "location" be assigned to a distributed vacuum mode?
4. The precise mechanism by which the Planck-scale wave cutoff operates (cf. footnote 4) is not understood (although there is no doubt that it must operate), and it may operate in such a way as to alter our spectral roughness estimates.

I had been crudely treating space and time as separate entities everywhere above, and pretending the universe-model box instantly "responds" (by shrinking or expanding) to changes in its contained vacuum energy. That view obviously is only approximately correct.

Nevertheless, it seems very difficult to dispute that some such kind of feedback control process *must* be happening, since the forces involved are so gigantic and they are all oriented in the right direction to force $|\Lambda|$ toward zero. See figure 3.

14.1 Review of the FRW$_\kappa$ cosmological model

To make this more clear, let me consider (my version of) the usual cosmological model – which I call the FRW$_\kappa$ model (FRW=Friedmann, Robertson, and Walker). This

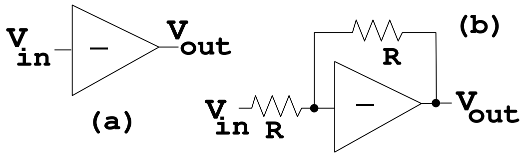

Figure 3: (a) A negating amplifier with $V_{\rm out} = -GV_{\rm in}$. If G is enormous and the amplifier has small time delay and good "high frequency response," then any attempt to make $|V_{\rm in}|$ small will fail to cause $|V_{\rm out}|$ to be small because tiny amounts of input noise will be hugely amplified. (b) If however, the same amplifier is connected in the *negative feedback* configuration shown, then $V_{\rm out} = -V_{\rm in}$ (and in particular if $V_{\rm in} = 0$, then $V_{\rm out} = 0$) accurately. Indeed the performance *improves* the larger G is and the smaller the time delay is, because the feedback works more quickly and accurately to cancel out the noise. Good commercial amplifiers have $G \approx 10^6$ and time delays $\approx 10^{-6}$second. But the feedback mechanism that (we propose) is operating to zero the cosmical constant Λ, has performance far superior to any commercial amplifier: $G \approx 10^{56}$ and time delays $< 10^{-60}$ second.

involves a model universe with a Lorentzian metric with infinitesimal line element ds given by

$$\mathrm{d}s^2 = A(t)^2 \mathrm{CCM}_\kappa - \mathrm{d}t^2 \quad (22)$$

where CCM_κ denotes any 3D purely spatial metric of constant curvature κ and $\kappa \in \{-1, 0, +1\}$. For example if $\kappa = +1$ one could use for CCM_κ the metric of the 3D surface of the unit sphere in $\mathbf{R}^4$, if $\kappa = 0$ one could use a "flat torus" (cube with periodic boundary conditions), and if $\kappa = -1$ one could use any compact hyperbolic manifold. In this metric there are 3 spatial coordinates and one timelike coordinate t, and the function $A(t)$ is the "size of the universe at time t." All the points in a slice t = constant of this metric are equivalent if the CCM metric has a transitive symmetry group (as it does in the case of the 3-sphere and flat torus).

Now the unique solution of the Einstein field equations of GR, augmented by a cosmical constant Λ, within this class of metrics and assuming certain equations of state are obeyed by matter and radiation, is known and is given in [46] (their EQ 27.40 page 730). It is the following specification for $A(t)$:

$$\int_0^A \frac{\sqrt{3}\,\mathrm{d}a}{\sqrt{\Lambda a^2 - 3\kappa + 8\pi\rho_{m0}A_0^3/a + 8\pi\rho_{r0}A_0^4/a^2}} = t. \quad (23)$$

Here everything is in Planck units, ρ_{m0} is the matter density now, ρ_{r0} is the radiation density now, A_0 is the size of the universe now, $\kappa \in \{-1, 0, 1\}$, and Λ is the cosmical constant, and we are solving for $A(t)$, the linear "size of the universe" at time t. The integral is an elliptic one and hence may be evaluated in closed form. For our purposes it suffices to know which one, among the 4 terms inside the square root, dominates the others. When the universe was about 4000 times smaller than it is now (and before that), the radiation term (the last term inside the square root) dominated and $A(t) \approx (32\pi\rho_{r0}/3)^{1/4} A_0 \sqrt{t}$. (Incidentally, note the infinite expansion rate when $t \to 0^+$.) Then the matter term (the 3rd term) took over, yielding $A(t) \approx (6\pi\rho_{m0})^{1/3} A_0 t^{2/3}$. In principle it would be possible for the curvature term κ to take over in the future (assuming $\kappa \neq 0$), which could lead, for example, if $\kappa = 1$, to a "turnaround" when the quantity inside the square root became zero, followed by a change of sign of the square root. Such a turnaround would ultimately be followed by a "big crunch." However, the recent supernova data [56] allegedly shows that is not going to happen, because we allegedly already (when the universe was about 30% smaller than it is now) have made the transition to a Λ-dominated era, with $\Lambda > 0$. If this is so, the universe has now passed "over the brink" and is beginning a dramatic final explosion with exponential growth behavior $A(t) \propto e^{\Lambda t/3}$. (Everything will get exponentially isolated from everything else, quite a boring fate.)

Consider the effect, at some time t, of spectral roughness suddenly increasing $|\Lambda|$ to 10^{56} times its present alleged value. The effects, according to EQ 23, would be dramatic – the universe would expand, or turnaround and shrink (depending on the sign of Λ), by a factor of 3 from its present size $A(t_{\rm now}) \approx 10^{10}$ light years, in $\Delta t \approx .5$ seconds. *Note that this, viewed naively, involves "travel" far faster than the speed of light*, which is really not forbidden by GR at all, since this is an exact solution of GR$\cup\Lambda$. Of course, we know the properties of spectral roughness are such that we really only need to expand or shrink the universe, not by a factor of 3, but instead only by a factor of $1 + 10^{-60}$ (or considerably less!), to force perfect cancellation and $\Lambda = 0$. This would only require 10^{-60} seconds (or much less – plausibly only 10^{-103} seconds).

The two morals of this little investigation are first, that feedback forces pulling Λ toward 0 are *vastly* larger than the effects of anything else in the universe, and second, that in principle it does not seem forbidden for the feedback loop to act on *extremely* tiny time scales, *mocking* naive speed of light delay times. It is of course the *combination* of feedback and roughness (and the perfect energy cancellation at energy scales larger than the roughness) that makes everything work, and (cf. figure 3) work well.

14.2 Important scales

When considering how feedback might operate, it is important to keep certain length, mass, and energy scales in mind.

The mass of the universe is about $m_{\rm univ} \approx 10^{52}$kg, so that the Compton wavelength of the universe is $\lambda_{\rm univ} = h/(m_{\rm univ}c) \approx 2 \times 10^{-94}$ meters.

The length scale of the universe is about 2×10^{10} light years, i.e. $\ell_{\rm univ} \approx 2 \times 10^{28}$ meters (and this is increasing with time).

The additive amount $\Delta\ell_{\rm univ}$ by which the universe must change its length scale, in order to cause $\Lambda = 0$, is either

1. If we use Tsang's fluctuation bound from §11.4 with $\epsilon = 0.01$, then $\Delta\ell_{\rm univ} < 4\ell_{\rm planck} \approx 6 \times 10^{-35}$ meter.

2. If we believe in the empirical findings of figure 1, then $\Delta\ell_{\rm univ} \approx 4(\ell_{\rm planck})^2/\ell_{\rm univ} \approx 5 \times 10^{-96}$ meters (and this is decreasing with time).

14.3 Expanding universe?

Yet another consideration helping to make feedback mysterious is the fact that we live in an expanding universe, as opposed to a constant-size box. Then presumably the feedback would have the effect (crudely described) of causing the universe's size to "hop" rather than increase smoothly, thus staying most of the time near the magic values. Of course really the dynamics would be quantum, not classical, so no deterministic value of the universe's "exact size" would ever be available, and hence really the universe's size could only be described by a complex probability amplitude wavefunction presumably peaked near the classical size value. Presumably the dynamics obeyed by that wavefunction would be similar to the "1D Anderson localization" [4] phenomenon for waves under the Schrödinger equation in a "random looking" 1D potential, except in our case the random potential would presumably be superposed on top of a "moving valley" or "tilt" of some kind serving to "guide" the universe's size along its correct expansion trajectory. Thus the dynamics in some respects might be quite similar to the mathematics of electronic conduction in amorphous solids [4].

15 Could quantum fuzziness explain small but nonzero Λ?

Now since in our box model universe, an adjustment of the box side length by $\Delta\ell_{\rm univ} \approx 10^{-95}$meters probably is sufficient to make $\Lambda = 0$, and since it has often been argued that in fact, due to fundamental "uncertainty principles," lengths are simply not *defined* to accuracies better than $\approx \pm\ell_{\rm planck} \approx 10^{-35}$meters, the universe might well be *unable* to accomplish forcing $\Lambda = 0$ *exactly,* since all its attempts would be defeated by quantum uncertainty.

On the other hand it could be argued that what matters is not any impossible "measurement" of the length of the box side, but rather, a measurement of the total mass-energy of all the zeropoint modes in the box, which may not be the same thing; or it could be argued that the universe's "size scale" for this purpose is really an average of a vast number of different length measurements and hence is in principle determined to far greater accuracy than $\pm\ell_{\rm planck}$, e.g. to $\pm\lambda_{\rm univ} \approx 2\times 10^{-94}$ meters. In that case it could be argued that quantum fuzziness only recently began to play a role (since only in the recent history of the universe did $\Delta\ell_{\rm univ}$, become comparably small or smaller than $\lambda_{\rm univ}$) and hence only recently was $|\Lambda|$ forced to rise much above 0.

I'm not sure what the net effect of all that would be.

A crude argument once suggested by R.Sorkin [59] to estimate this is as follows. Sorkin argued that Λ and the 4D volume V of the universe should be "canonically conjugate" quantum quantities (this is basically just the usual position-momentum and energy-time uncertainty principles), hence should obey an uncertainty relation $\Delta\Lambda\Delta V \gtrsim 1$ (in Planck units); and also he argues that $\Delta V \approx \sqrt{V}$. "Hence," whatever process is making $|\Lambda|$ small should be unable to defeat the uncertainty principle, causing $\Lambda \approx 1/\sqrt{V}$ in Planck units. Sure enough, using the current age and "size" of the universe, e.g. $V \approx (10^{10}\text{lightyear})^3 \times 14 \times 10^9\text{year}$, i.e. $V \approx 1.6 \times 10^{243}$ in Planck units, we find $\Lambda \approx 2.5\times 10^{-122}$ in Planck units, i.e. $\Lambda \approx 9.4\times 10^{-53}\text{meter}^{-2}$, which indeed seems to have the right order of magnitude.

This could explain why Λ is small but nonzero (as suggested by recent supernovae work [56], which indicates a repulsive Λ) with indeed roughly the correct magnitude. I think, however, that this Sorkin idea is very crude and more understanding would be gained by trying to model the quantum dynamics of such feedback systems, complete with various kinds of "spectral roughness" and "Anderson localization," directly.

That task is left for future authors.

16 Conclusion and open problems

I feel that this paper has provided all the ingredients necessary to resolve the cosmical constant crisis – and in a fully satisfactory way, without needing to model the creation of the universe, depend on "anthropic principles," or pass the buck to other mysterious problems. Namely, the key ingredients are:

1. The zeropoint energy of fermion modes is negative and hence in principle could cancel boson zeropoint energies (§2).

2. Supersymmetry forces there to be a 1-to-1 correspondence between boson and fermion modes, which indeed causes such a cancellation at first order in an asymptotic expansion in the high mode-cutoff energy (§3; I am here assuming it's the Planck energy, although any very large energy could have been used).

3. More importantly, even in highly general "broken" supersymmetric theories, an identity forces the cancelation to hold at *second* order in the asymptotic series (§4-5).

4. The third and higher order terms in the series do not exist due to "spectral roughness" (§10-12).

5. This roughness forces there to be (if the universe is modeled as a box with periodic boundary conditions and *if* the box-size happens to lie on a particular, very dense, set of "magic values") *exact* cancelation yielding $\Lambda = 0$.

6. 50% of the "magic" values would be "stable" in the sense that, if the universe's size were *not* in the stable magic subset, enormous "feedback" forces would then act extremely quickly to perturb the size of the universe by the tiny amount necessary to bring it there (§13-14).

7. Quantum fuzziness might prevent this feedback process from *precisely* achieving $\Lambda = 0$ so it might be that Λ could come out small but nonzero. The dynamics of this feedback, if treated quantum mechanically, might resemble Anderson localization for amorphous conductors (§15).

Although all the ingredients are there, and all of them seem logically inescapable, some of them need more fleshing out.

First, spectral roughness needs to be better understood and in more "generic" model universes. For example, a good start would be to understand it in a model universe which was a generic parallelipiped (i.e. with incommensurable irrational corner coordinates) with periodic boundary conditions, instead of just a cube.

More generally we want to understand spectral roughness on general Riemannian manifolds (and also, the universe is a *Lorentzian* manifold – what should be done about that?), which leads to the problem – also heavily studied – of the behavior of the error in Weyl's asymptotic spectral formula. And: To what extent may "quantum chaos" notions be made rigorous?

In both cases we are interested not in *counting* eigenfrequencies, but rather in *summing* them, so all previous studies would need to be altered.

One angle from which it might be productive to view spectral roughness would be asking what the effect of various kinds of smoothing processes would be on it. For example, instead of smoothing by means of averaging over a width-W "moving window," another approach would be to convolve with a Gaussian. This would have the advantage that it would get rid of jump discontinuities, as opposed to merely shrinking them. Still, the following argument is pretty convincing that enough roughness still would remain (even if the width of the Gaussian were an arbitrary power of U) to invalidate any asymptotic series beyond some term. Consider, instead of the integer 3-vector *points* in a ball of radius U, Gaussian density blobs. It is fairly easy to convince oneself that the total mass inside our radius-U ball as a function of U will still exhibit "fluctuations" of height at least some positive power of U.

Second, the accuracy to which EQ 16 is valid in supersymmetric theories needs to be analysed more precisely.

Third, the precise dynamics of the "feedback process" need to be modeled and/or understood in far greater detail, especially if we want to understand its inherent limitations and thus the slightly *nonzero* value of the cosmical constant. There are many mysterious complicating factors – such as, how will this work both locally and globally in curved spacetime?

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